

**$tt^*$ -GEOMETRY IN QUANTUM COHOMOLOGY**

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ABSTRACT. We study possible real structures in the space of solutions to the quantum differential equation. We show that, under mild conditions, a real structure in orbifold quantum cohomology yields a pure and polarized  $tt^*$ -geometry near the large radius limit. We compute an example of  $\mathbb{P}^1$  which is pure and polarized over the whole Kähler moduli space  $H^2(\mathbb{P}^1, \mathbb{C}^*)$ .

## 1. INTRODUCTION

The quantum cohomology is a family  $(H^*(X), \circ_\tau)$  of commutative rings parametrized by  $\tau \in H^*(X)$  and satisfies the following “integrability”: A family  $\nabla^z$  of connections (Dubrovin connection) on the trivial vector bundle  $H^*(X) \times H^*(X) \rightarrow H^*(X)$

$$\nabla^z = d + \frac{1}{z} \sum_{i=1}^N (\phi_i \circ_\tau) dt^i, \quad z \in \mathbb{C}^*$$

is flat for all  $z \in \mathbb{C}^*$ . Here we take a basis  $\{\phi_i\}_{i=1}^N$  of  $H^*(X)$  and linear co-ordinates  $\{t^i\}_{i=1}^N$  dual to it and  $\tau = \sum_{i=1}^N t^i \phi_i$ . The Dubrovin connection  $\nabla^z$  is extended to a flat connection  $\hat{\nabla}$  over  $\{(\tau, z) \in H^*(X) \times \mathbb{C}^*\}$  and this defines a local system  $R$  of  $\mathbb{C}$ -vector spaces over  $H^*(X) \times \mathbb{C}^*$ . In this paper, we consider its real structure — a sub local system  $R_{\mathbb{R}} \subset R$  of  $\mathbb{R}$ -vector spaces. When written in a frame compatible with a real structure, the holomorphic connection  $\nabla^z$  gains antiholomorphic part and gives rise to  $tt^*$ -geometry or topological–anti-topological fusion [5, 16, 21].

The study of real structures in quantum cohomology is motivated from mirror symmetry. The quantum cohomology of a Calabi-Yau threefold  $X$  defines a variation of Hodge structure (VHS) over  $H^{1,1}(X)$  [32, 13]

$$F^3 \subset F^2 \subset F^1 \subset F^0 = H^{*,*}(X), \quad F^p = \bigoplus_{k \geq p} H^{3-k, 3-k}(X)$$

by the Dubrovin connection  $\nabla^z$ . Mirror symmetry conjecture says that this is isomorphic to the VHS  $\check{F}^p = \bigoplus_{k \geq p} H^{k, 3-k}(Y)$  of a mirror Calabi-Yau  $Y$  over the complex moduli space of  $Y$ . While the VHS of  $Y$  has a natural real structure  $H^3(Y, \mathbb{R})$  (and an integral structure  $H^3(Y, \mathbb{Z})$ ), the VHS associated to the quantum cohomology of  $X$  does not seem to have a natural real structure. In the companion paper [24], we studied mirror symmetry for toric orbifolds. The calculation there suggested that the  $K$ -theory and the  $\hat{\Gamma}$ -class of  $X$  define a natural integral (hence real) structure on the quantum cohomology VHS. The same rational structure was also proposed by Katzarkov-Kontsevich-Pantev [26] independently.

In this paper, we use the language of *semi-infinite variation of Hodge structure* (henceforth  $\frac{\infty}{2}$ VHS) due to Barannikov [3, 4] to deal also with non Calabi-Yau case. Here we briefly explain the  $\frac{\infty}{2}$ VHS of quantum cohomology. Let  $L(\tau, z)$  be the fundamental solution for Dubrovin-flat sections  $\nabla^z s = 0$ :

$$L: H^*(X) \times \mathbb{C}^* \rightarrow \text{End}(H^*(X)), \quad \nabla^z L(\tau, z)\phi = 0, \quad \phi \in H^*(X),$$

which is explicitly given by the gravitational descendants (see (29)). Following Coates-Givental [11], we introduce an infinite dimensional vector space  $\mathcal{H}^X$  by

$$\mathcal{H}^X := H^*(X) \otimes \mathcal{O}(\mathbb{C}^*)$$

where  $\mathcal{O}(\mathbb{C}^*)$  denotes the space of holomorphic functions on  $\mathbb{C}^*$  with co-ordinate  $z$ . We identify  $\mathcal{H}^X$  with the space of  $\nabla^z$ -flat sections by the map  $\mathcal{H}^X \ni \alpha(z) \mapsto L(\tau, z)\alpha(z)$ . We define the family  $\mathbb{F}_\tau$  of “semi-infinite” subspaces of  $\mathcal{H}^X$  as

$$\mathbb{F}_\tau := L(\tau, z)^{-1}(H^*(X) \otimes \mathcal{O}(\mathbb{C})) \subset \mathcal{H}^X, \quad \tau \in H^*(X),$$

where  $\mathcal{O}(\mathbb{C})$  denotes the space of holomorphic functions on  $\mathbb{C}$ . The semi-infinite flag  $\cdots \subset z^{-1}\mathbb{F}_\tau \subset \mathbb{F}_\tau \subset z\mathbb{F}_\tau \subset \cdots$  satisfies properties analogous to the usual finite dimensional VHS:

- (1)  $\frac{\partial}{\partial t^i} \mathbb{F}_\tau \subset z^{-1}\mathbb{F}_\tau$  (Griffiths Transversality)
- (2)  $(\mathbb{F}_\tau, \mathbb{F}_\tau)_{\mathcal{H}^X} \subset \mathcal{O}(\mathbb{C})$  (Bilinear Relations)

where the pairing  $(\cdot, \cdot)_{\mathcal{H}^X}$  is defined by  $(\alpha, \beta)_{\mathcal{H}^X} = \int_X \alpha(-z) \cup \beta(z)$  for  $\alpha, \beta \in \mathcal{H}^X$ . We call this (the moving subspace realization of) the *quantum cohomology  $\frac{\infty}{2}$ VHS*.

A real structure of the quantum cohomology (a sub  $\mathbb{R}$ -local system  $R_{\mathbb{R}} \subset \widehat{\nabla}$ ) induces a subspace  $\mathcal{H}_{\mathbb{R}}^X$  of  $\mathcal{H}^X$

$$\mathcal{H}_{\mathbb{R}}^X := \{\alpha(z) \in \mathcal{H}^X; L(\tau, z)\alpha(z) \in R_{\mathbb{R},(\tau,z)} \text{ when } |z| = 1\}$$

and the involution  $\kappa_{\mathcal{H}}: \mathcal{H}^X \rightarrow \mathcal{H}^X$  fixing  $\mathcal{H}_{\mathbb{R}}^X$  and satisfying  $\kappa_{\mathcal{H}}(f(z)\alpha) = \overline{f(1/\bar{z})}\kappa_{\mathcal{H}}(\alpha)$  for  $f(z) \in \mathcal{O}(\mathbb{C}^*)$ . For a “good” real structure, we hope the following properties:

- (3)  $\mathbb{F}_\tau \oplus z^{-1}\kappa_{\mathcal{H}}(\mathbb{F}_\tau) = \mathcal{H}^X$ , (Hodge Decomposition)
- (4)  $(\kappa_{\mathcal{H}}(\alpha), \alpha)_{\mathcal{H}^X} > 0$ ,  $\alpha \in \mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) \setminus \{0\}$ . (Bilinear Inequality)

When  $X$  is Calabi-Yau, these (1), (2), (3), (4) are translations of the corresponding properties for the finite dimensional VHS. The properties (3) and (4) are called *pure* and *polarized* respectively. Our main theorem states that (3), (4) indeed hold near the “large radius limit” i.e.  $\tau = -x\omega$ ,  $\Re(x) \rightarrow \infty$  for some Kähler class  $\omega$ , under mild assumptions on the real structures:

**Theorem 1.1** (Theorem 3.9). *Assume that a real structure is invariant under the monodromy (Galois) transformations given by  $G^{\mathcal{H}}(\xi), \xi \in H^2(X, \mathbb{Z})$  (see (39) and Proposition 3.7). If the condition (47) (which is empty when  $X$  is a manifold) holds,  $\mathbb{F}_\tau$  is pure (3) near the large radius limit. If moreover the condition (49) holds and  $H^*(X) = \bigoplus_p H^{p,p}(X)$ ,  $\mathbb{F}_\tau$  is polarized (4) near the large radius limit.*

In the theorem above, we allow  $X$  to be an orbifold or a smooth Deligne-Mumford stack. See Theorem 3.9 for a more precise statement. Given a real structure satisfying (3) and (4), quantum cohomology gives a Hermitian vector bundle with a connection  $D$  and endomorphisms  $\kappa, C, \tilde{C}, \mathcal{U}, \mathcal{Q}$  satisfying the  $tt^*$ -equations (Proposition 2.13). This structure ( $tt^*$ -geometry) was discovered by Cecotti-Vafa [5, 8] and has been studied by Dubrovin [16] and Hertling [21].  $tt^*$ -geometry also gives an example of a harmonic bundle or a twistor structure of Simpson [36]. Closely related results have been shown in a more abstract setting for TERP structures in [21, 22]. In fact, when  $X$  is Fano and the Kähler class  $\omega$  is  $c_1(X)$ , the conclusions of Theorem 1.1 can be deduced from [22, Theorem 7.3].

This paper is structured as follows. In Section 2, we introduce real structures for a general graded  $\frac{\infty}{2}$ VHS. This section is a translation of the work of Hertling [21] in terms of Barannikov's semi-infinite Hodge structure. In Section 3, we study real structures in (orbifold) quantum cohomology and prove Theorem 1.1. We also give a review of the  $\widehat{\Gamma}$ -real structure given by  $K$ -theory [23, 24, 26] and see that this real structure satisfies the assumptions in Theorem 1.1. In Section 4, we calculate an example of  $tt^*$ -geometry for  $X = \mathbb{P}^1$  with respect to the  $\widehat{\Gamma}$ -real structure. In this case, by Sabbah [34], the  $tt^*$ -geometry is pure and polarized over the whole Kähler moduli space  $H^2(\mathbb{P}^1, \mathbb{C}^*)$ . We confirm Cecotti-Vafa's calculation [6] by a recursive Birkhoff factorization.

The convergence of the quantum cohomology is assumed throughout the paper. Also we consider only the even parity part of the cohomology, *i.e.*  $H^*(X)$  means  $\bigoplus_k H^{2k}(X)$ . Note that the orbifold cohomology  $H_{\text{orb}}^*(\mathcal{X})$  is denoted also by  $H_{\text{CR}}^*(\mathcal{X})$  in the literature.

This paper is a revision of part of the preprint [23] concerning real structures. The integral structure part of [23] was separated in [24].

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## 2. REAL STRUCTURES ON $\frac{\infty}{2}$ VHS

We introduce real (and integral) structures for a general semi-infinite variation of Hodge structure or  $\frac{\infty}{2}$ VHS. We explain that a  $\frac{\infty}{2}$ VHS with a real structure yields a Cecotti-Vafa structure when it is pure. A  $\frac{\infty}{2}$ VHS was originally introduced by Barannikov [3, 4]. A  $\frac{\infty}{2}$ VHS with a real structure considered here corresponds to the TERP structure due to Hertling [21] (see Remark 2.3). The exposition here largely follows the line of [21, 12].

**2.1. Definition.** Let  $\mathcal{M}$  be a smooth complex analytic space and  $\mathcal{O}_{\mathcal{M}}$  be the analytic structure sheaf on  $\mathcal{M}$ . We introduce an additional complex plane  $\mathbb{C}$  with co-ordinate  $z$

and consider the product  $\mathcal{M} \times \mathbb{C}$ . Let  $\pi: \mathcal{M} \times \mathbb{C} \rightarrow \mathcal{M}$  be the projection. A  $\frac{\infty}{2}$ VHS is a module over the push-forward  $\pi_*\mathcal{O}_{\mathcal{M} \times \mathbb{C}}$  of the analytic structure sheaf on  $\mathcal{M} \times \mathbb{C}$ . Let  $\Omega_{\mathcal{M}}^1$  be the sheaf of holomorphic 1-forms on  $\mathcal{M}$  and  $\Theta_{\mathcal{M}}$  be the sheaf of holomorphic tangent vector fields on  $\mathcal{M}$ .

**Definition 2.1** ([12]). A *semi-infinite variation of Hodge structures*, or  $\frac{\infty}{2}$ VHS is a locally free  $\pi_*\mathcal{O}_{\mathcal{M} \times \mathbb{C}}$ -module  $\mathcal{F}$  of rank  $N$  endowed with a holomorphic flat connection

$$\nabla: \mathcal{F} \rightarrow z^{-1}\mathcal{F} \otimes \Omega_{\mathcal{M}}^1$$

and a perfect pairing

$$(\cdot, \cdot)_{\mathcal{F}}: \mathcal{F} \times \mathcal{F} \rightarrow \pi_*\mathcal{O}_{\mathcal{M} \times \mathbb{C}}$$

satisfying

$$\begin{aligned} \nabla_X(fs) &= (Xf)s + f\nabla_X s, \\ [\nabla_X, \nabla_Y]s &= \nabla_{[X, Y]}s, \\ (s_1, f(z)s_2)_{\mathcal{F}} &= (f(-z)s_1, s_2)_{\mathcal{F}} = f(z)(s_1, s_2)_{\mathcal{F}}, \\ (s_1, s_2)_{\mathcal{F}} &= (s_2, s_1)_{\mathcal{F}}|_{z \rightarrow -z}, \\ X(s_1, s_2)_{\mathcal{F}} &= (\nabla_X s_1, s_2)_{\mathcal{F}} + (s_1, \nabla_X s_2)_{\mathcal{F}} \end{aligned}$$

for sections  $s, s_1, s_2$  of  $\mathcal{F}$ ,  $f \in \pi_*\mathcal{O}_{\mathcal{M} \times \mathbb{C}}$  and  $X, Y \in \Theta_{\mathcal{M}}$ . Here,  $\nabla_X$  is a map from  $\mathcal{F}$  to  $z^{-1}\mathcal{F}$  and  $z^{-1}\mathcal{F}$  is regarded as a submodule of  $\mathcal{F} \otimes_{\pi_*\mathcal{O}_{\mathcal{M} \times \mathbb{C}}} \pi_*\mathcal{O}_{\mathcal{M} \times \mathbb{C}}^*$ . The first two properties are part of the definition of a flat connection. The pairing  $(\cdot, \cdot)_{\mathcal{F}}$  is perfect in the sense that it induces an isomorphism of the fiber  $\mathcal{F}_{\tau}$  at  $\tau \in \mathcal{M}$  with  $\text{Hom}_{\mathcal{O}(\mathbb{C})}(\mathcal{F}_{\tau}, \mathcal{O}(\mathbb{C}))$ , where  $\mathcal{O}(\mathbb{C})$  is the space of holomorphic functions on  $\mathbb{C}$ .

A *graded  $\frac{\infty}{2}$ VHS* is a  $\frac{\infty}{2}$ VHS  $\mathcal{F}$  endowed with a  $\mathbb{C}$ -endomorphism  $\text{Gr}: \mathcal{F} \rightarrow \mathcal{F}$  and an Euler vector field  $E \in H^0(\mathcal{M}, \Theta_{\mathcal{M}})$  satisfying

$$\begin{aligned} \text{Gr}(fs_1) &= (2(z\partial_z + E)f)s_1 + f\text{Gr}(s_1), \\ [\text{Gr}, \nabla_X] &= \nabla_{2[E, X]}, \quad X \in \Theta_{\mathcal{M}}, \\ 2(z\partial_z + E)(s_1, s_2)_{\mathcal{F}} &= (\text{Gr}(s_1), s_2)_{\mathcal{F}} + (s_1, \text{Gr}(s_2))_{\mathcal{F}} - 2n(s_1, s_2)_{\mathcal{F}} \end{aligned}$$

where  $n \in \mathbb{C}$ . □

A  $\frac{\infty}{2}$ VHS is a semi-infinite analogue of the usual finite dimensional VHS without a real structure. The “semi-infinite” flag  $\cdots \subset z\mathcal{F} \subset \mathcal{F} \subset z^{-1}\mathcal{F} \subset z^{-2}\mathcal{F} \subset \cdots$  plays the role of the Hodge filtration. The flat connection  $\nabla_X$  shifts this filtration by one — this is an analogue of the Griffiths transversality.

The structure of a graded  $\frac{\infty}{2}$ VHS  $\mathcal{F}$  can be rephrased in terms of a locally free sheaf  $\mathcal{R}^{(0)}$  over  $\mathcal{M} \times \mathbb{C}$  with a flat connection  $\widehat{\nabla}$ . Here  $\mathcal{R}^{(0)}$  is a locally free  $\mathcal{O}_{\mathcal{M} \times \mathbb{C}}$ -module of rank  $N$  such that  $\mathcal{F} = \pi_*\mathcal{R}^{(0)}$ . We define the meromorphic connection<sup>1</sup>  $\widehat{\nabla}$  on  $\mathcal{R}^{(0)}$

$$\widehat{\nabla}: \mathcal{R}^{(0)} \longrightarrow \frac{1}{z}\mathcal{R}^{(0)} \otimes \left( \pi^*\Omega_{\mathcal{M}}^1 \oplus \mathcal{O}_{\mathcal{M} \times \mathbb{C}} \frac{dz}{z} \right)$$

by the formula

$$(5) \quad \widehat{\nabla}s := \nabla s + \left( \frac{1}{2}\text{Gr}(s) - \nabla_E s - \frac{n}{2}s \right) \frac{dz}{z}$$

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<sup>1</sup>The extended connection  $\widehat{\nabla}$  over  $\mathcal{M} \times \mathbb{C}$  was denoted by  $\nabla$  in the companion paper [24].

for a section  $s$  of  $\mathcal{F} = \pi_* \mathcal{R}^{(0)}$ . It is easy to see that the conditions on  $\text{Gr}$  and  $\nabla$  above imply that  $\widehat{\nabla}$  is also flat. The pairing  $(\cdot, \cdot)_{\mathcal{F}}$  on  $\mathcal{F}$  induces a non-degenerate pairing on  $\mathcal{R}^{(0)}$ :

$$(\cdot, \cdot)_{\mathcal{R}^{(0)}} : (-)^* \mathcal{R}^{(0)} \otimes \mathcal{R}^{(0)} \rightarrow \mathcal{O}_{\mathcal{M} \times \mathbb{C}},$$

where  $(-): \mathcal{M} \times \mathbb{C} \rightarrow \mathcal{M} \times \mathbb{C}$  is a map  $(\tau, z) \mapsto (\tau, -z)$ . This pairing is flat with respect to  $\widehat{\nabla}$  on  $\mathcal{R}^{(0)}$  and  $(-)^* \widehat{\nabla}$  on  $(-)^* \mathcal{R}^{(0)}$ . Denote by  $\mathcal{R}$  the restriction of  $\mathcal{R}^{(0)}$  to  $\mathcal{M} \times \mathbb{C}^*$ . Since  $\widehat{\nabla}$  is regular outside  $z = 0$ ,  $\mathcal{R}$  is a flat vector bundle over  $\mathcal{M} \times \mathbb{C}^*$ . Let  $R \rightarrow \mathcal{M} \times \mathbb{C}^*$  be the  $\mathbb{C}$ -local system underlying the flat vector bundle  $\mathcal{R}$ . This has a pairing  $(\cdot, \cdot)_R : (-)^* R \otimes_{\mathbb{C}} R \rightarrow \mathbb{C}$  induced from  $(\cdot, \cdot)_{\mathcal{R}^{(0)}}$ .

**Definition 2.2.** Let  $\mathcal{F}$  be a graded  $\frac{\infty}{2}$ VHS with  $n \in \mathbb{Z}$ . A *real structure* on  $\frac{\infty}{2}$ VHS is a sub  $\mathbb{R}$ -local system  $R_{\mathbb{R}} \rightarrow \mathcal{M} \times \mathbb{C}^*$  of  $R$  such that  $R = R_{\mathbb{R}} \oplus \mathbf{i} R_{\mathbb{R}}$  and the pairing takes values in  $\mathbb{R}$  on  $R_{\mathbb{R}}$

$$(\cdot, \cdot)_R : (-)^* R_{\mathbb{R}} \otimes_{\mathbb{R}} R_{\mathbb{R}} \rightarrow \mathbb{R}.$$

An *integral structure* on  $\frac{\infty}{2}$ VHS is a sub  $\mathbb{Z}$ -local system  $R_{\mathbb{Z}} \rightarrow \mathcal{M} \times \mathbb{C}^*$  of  $R$  such that  $R = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$  and the pairing takes values in  $\mathbb{Z}$  on  $R_{\mathbb{Z}}$

$$(\cdot, \cdot)_R : (-)^* R_{\mathbb{Z}} \otimes R_{\mathbb{Z}} \rightarrow \mathbb{Z}$$

and is unimodular *i.e.* induces an isomorphism  $R_{\mathbb{Z}, (\tau, -z)} \cong \text{Hom}(R_{\mathbb{Z}, (\tau, z)}, \mathbb{Z})$  for  $(\tau, z) \in \mathcal{M} \times \mathbb{C}^*$ .  $\square$

**Remark 2.3.** A graded  $\frac{\infty}{2}$ VHS with a real structure defined here is almost equivalent to a  $\text{TERP}(n)$  structure introduced by Hertling [21]. The only difference is that the flat connection  $\widehat{\nabla}$  in  $\text{TERP}(n)$  structure is not assumed to arise from a grading operator  $\text{Gr}$  and an Euler vector field  $E$ . Therefore, a graded  $\frac{\infty}{2}$ VHS gives a  $\text{TERP}$  structure, but the converse is not true in general. For the convenience of the reader, we give differences in convention between [21] and us. Let  $\tilde{\nabla}$ ,  $\tilde{R}$ ,  $\tilde{R}_{\mathbb{R}}$ ,  $\tilde{P}: \tilde{R} \otimes (-)^* \tilde{R} \rightarrow \mathbb{C}$  denote the flat connection,  $\mathbb{C}$ -local system, sub  $\mathbb{R}$ -local system and a pairing appearing in [21]. They are related to our  $\widehat{\nabla}$ ,  $R$ ,  $R_{\mathbb{R}}$ ,  $(\cdot, \cdot)_{\mathcal{R}^{(0)}}$  as

$$\begin{aligned} \tilde{\nabla} &= \widehat{\nabla} + \frac{n}{2} \frac{dz}{z}, \\ \tilde{R} &= (-z)^{-\frac{n}{2}} R, \quad \tilde{R}_{\mathbb{R}} = (-z)^{-\frac{n}{2}} R_{\mathbb{R}}, \\ \tilde{P}(s_1, s_2) &= z^n (s_2, s_1)_{\mathcal{R}^{(0)}}. \end{aligned}$$

Then  $\tilde{R}$  is the local system defined by  $\tilde{\nabla}$ ,  $\tilde{P}$  is  $\tilde{\nabla}$ -flat and

$$\tilde{P}(\tilde{R}_{\mathbb{R}, (\tau, z)} \times \tilde{R}_{\mathbb{R}, (\tau, -z)}) = z^n \left( z^{-n/2} R_{\mathbb{R}, (\tau, -z)}, (-z)^{-n/2} R_{\mathbb{R}, (\tau, z)} \right)_R \subset \mathbf{i}^n \mathbb{R}.$$

## 2.2. Semi-infinite period map.

**Definition 2.4.** For a graded  $\frac{\infty}{2}$ VHS  $\mathcal{F}$ , the *spaces*  $\mathcal{H}$ ,  $\mathcal{V}$  of *multi-valued flat sections* are defined to be

$$\begin{aligned} \mathcal{H} &:= \{s \in \Gamma(\widetilde{\mathcal{M}} \times \mathbb{C}^*, \mathcal{R}) ; \nabla s = 0\}, \\ \mathcal{V} &:= \{s \in \Gamma((\mathcal{M} \times \mathbb{C}^*)^\sim, \mathcal{R}) ; \widehat{\nabla} s = 0\}, \end{aligned}$$

where  $\widetilde{\mathcal{M}}$  and  $(\mathcal{M} \times \mathbb{C}^*)^\sim$  are the universal covers of  $\mathcal{M}$  and  $\mathcal{M} \times \mathbb{C}^*$  respectively. The space  $\mathcal{H}$  is a free  $\mathcal{O}(\mathbb{C}^*)$ -module, where  $\mathcal{O}(\mathbb{C}^*)$  is the space of holomorphic functions on  $\mathbb{C}^*$ . The space  $\mathcal{V}$  is a finite dimensional  $\mathbb{C}$ -vector space identified with the fiber of the local system  $R$ . The flat connection  $\widehat{\nabla}$  and the pairing  $(\cdot, \cdot)_{\mathcal{R}^{(0)}}$  on  $\mathcal{R}^{(0)}$  induce an operator

$$\widehat{\nabla}_{z\partial_z} : \mathcal{H} \rightarrow \mathcal{H}$$

and a pairing

$$(\cdot, \cdot)_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{O}(\mathbb{C}^*)$$

satisfying

$$\begin{aligned} (f(-z)s_1, s_2)_{\mathcal{H}} &= (s_1, f(z)s_2)_{\mathcal{H}} = f(z)(s_1, s_2)_{\mathcal{H}} \quad f(z) \in \mathcal{O}(\mathbb{C}^*), \\ (s_1, s_2)_{\mathcal{H}} &= (s_2, s_1)_{\mathcal{H}}|_{z \mapsto -z} \\ z\partial_z(s_1, s_2)_{\mathcal{H}} &= (\widehat{\nabla}_{z\partial_z}s_1, s_2)_{\mathcal{H}} + (s_1, \widehat{\nabla}_{z\partial_z}s_2)_{\mathcal{H}}. \end{aligned}$$

We regard the free  $\mathcal{O}(\mathbb{C}^*)$ -module  $\mathcal{H}$  with the operator  $\widehat{\nabla}_{z\partial_z}$  as a holomorphic flat vector bundle  $(H, \widehat{\nabla}_{z\partial_z})$  over  $\mathbb{C}^*$ :

$$(6) \quad H \rightarrow \mathbb{C}^*, \quad \mathcal{H} = \Gamma(\mathbb{C}^*, \mathcal{O}(H)).$$

Then  $\mathcal{V}$  can be identified with the space of multi-valued flat sections of  $H$ . A pairing  $(\cdot, \cdot)_{\mathcal{V}} : \mathcal{V} \otimes_{\mathbb{C}} \mathcal{V} \rightarrow \mathbb{C}$  is defined by

$$(7) \quad (s_1, s_2)_{\mathcal{V}} := (s_1(\tau, e^{\pi \mathbf{i}} z), s_2(\tau, z))_R$$

where  $s_1(\tau, e^{\pi \mathbf{i}} z) \in \mathcal{R}_{(\tau, -z)}$  denote the parallel translation of  $s_1(\tau, z) \in \mathcal{R}_{(\tau, z)}$  along the counterclockwise path  $[0, 1] \ni \theta \mapsto e^{\pi \mathbf{i} \theta} z$ .  $\square$

A  $\frac{\infty}{2}$ VHS  $\mathcal{F}$  on  $\mathcal{M}$  defines a map from  $\widetilde{\mathcal{M}}$  to the *Segal-Wilson Grassmannian* of  $\mathcal{H}$ . For  $u \in \mathcal{F}_{\tau}$  at  $\tau \in \widetilde{\mathcal{M}}$ , there exists a unique flat section  $s_u \in \mathcal{H}$  such that  $s_u(\tau) = u$ . This defines an embedding of a fiber  $\mathcal{F}_{\tau}$  into  $\mathcal{H}$ :

$$(8) \quad \mathbb{J}_{\tau} : \mathcal{F}_{\tau} \longrightarrow \mathcal{H}, \quad u \longmapsto s_u, \quad \tau \in \widetilde{\mathcal{M}}.$$

We call the image  $\mathbb{F}_{\tau} \subset \mathcal{H}$  of this embedding the *semi-infinite Hodge structure*. This is a free  $\mathcal{O}(\mathbb{C})$ -module of rank  $N$ . The family  $\{\mathbb{F}_{\tau} \subset \mathcal{H}\}_{\tau \in \widetilde{\mathcal{M}}}$  of subspaces gives the *moving subspace realization of  $\frac{\infty}{2}$ VHS*. Fix a  $\mathcal{O}(\mathbb{C}^*)$ -basis  $e_1, \dots, e_N$  of  $\mathcal{H}$ . Then the image of a local frame  $s_1, \dots, s_N$  of  $\mathcal{F}$  over  $\pi_*\mathcal{O}_{\mathcal{M} \times \mathbb{C}}$  under  $\mathbb{J}_{\tau}$  can be written as  $\mathbb{J}_{\tau}(s_j) = \sum_{i=1}^N e_i J_{ij}(\tau, z)$ . When  $z$  is restricted to  $S^1 = \{|z| = 1\}$ , the  $N \times N$  matrix  $(J_{ij}(\tau, z))$  defines an element of the smooth loop group  $LGL_N(\mathbb{C}) = C^{\infty}(S^1, GL_N(\mathbb{C}))$ . Another choice of a local basis of  $\mathcal{F}$  changes the matrix  $(J_{ij}(\tau, z))$  by right multiplication by a matrix with entries in  $\mathcal{O}(\mathbb{C})$ . Thus the Hodge structure  $\mathbb{F}_{\tau}$  gives a point  $(J_{ij}(\tau, z))_{ij}$  in the smooth Segal-Wilson Grassmannian  $\text{Gr}_{\frac{\infty}{2}}(\mathcal{H}) := LGL_N(\mathbb{C})/L^+GL_N(\mathbb{C})$  [33]. Here  $L^+GL_N(\mathbb{C})$  consists of smooth loops which are the boundary values of holomorphic maps  $\{z \in \mathbb{C} ; |z| < 1\} \rightarrow GL_N(\mathbb{C})$ . The map

$$\widetilde{\mathcal{M}} \ni \tau \longmapsto \mathbb{F}_{\tau} \in \text{Gr}_{\frac{\infty}{2}}(\mathcal{H})$$

is called the *semi-infinite period map*.

**Proposition 2.5** ([12, Proposition 2.9]). *The semi-infinite period map  $\tau \mapsto \mathbb{F}_\tau$  satisfies:*

- (i)  $X\mathbb{F}_\tau \subset z^{-1}\mathbb{F}_\tau$  for  $X \in \Theta_{\mathcal{M}}$ ;
- (ii)  $(\mathbb{F}_\tau, \mathbb{F}_\tau)_{\mathcal{H}} \subset \mathcal{O}(\mathbb{C})$ ;
- (iii)  $(\widehat{\nabla}_{z\partial_z} + E)\mathbb{F}_\tau \subset \mathbb{F}_\tau$ . In particular,  $\widehat{\nabla}_{z\partial_z}\mathbb{F}_\tau \subset z^{-1}\mathbb{F}_\tau$ .

The first property (ii) is an analogue of Griffiths transversality and the second (iii) is the Hodge-Riemann bilinear relation.

In terms of the flat vector bundle  $\mathbf{H} \rightarrow \mathbb{C}^*$  (6) (such that  $\mathcal{H} = \Gamma(\mathbb{C}^*, \mathcal{O}(\mathbf{H}))$ ), the Hodge structure  $\mathbb{F}_\tau \subset \mathcal{H}$  is considered to be an extension of  $\mathbf{H}$  to  $\mathbb{C}$  such that the flat connection has a pole of Poincaré rank 1 at  $z = 0$ .

Real and integral structures on  $\frac{\infty}{2}\text{VHS}$  define the following subspaces  $\mathcal{H}_{\mathbb{R}}, \mathcal{V}_{\mathbb{R}}, \mathcal{V}_{\mathbb{Z}}$ :

$$(9) \quad \begin{aligned} \mathcal{H}_{\mathbb{R}} &:= \{s \in \mathcal{H} ; s(\tau, z) \in R_{\mathbb{R},(\tau,z)} \text{ for } \tau \in \widetilde{\mathcal{M}} \text{ and } |z| = 1\} \\ \mathcal{V}_{\mathbb{R}} &:= \{s \in \mathcal{V} ; s(\tau, z) \in R_{\mathbb{R},(\tau,z)} \text{ for } (\tau, z) \in (\mathcal{M} \times \mathbb{C}^*)^\sim\} \\ \mathcal{V}_{\mathbb{Z}} &:= \{s \in \mathcal{V} ; s(\tau, z) \in R_{\mathbb{Z},(\tau,z)} \text{ for } (\tau, z) \in (\mathcal{M} \times \mathbb{C}^*)^\sim\} \end{aligned}$$

Then  $\mathcal{H}_{\mathbb{R}}$  becomes a (not necessarily free) module over the ring  $C^h(S^1, \mathbb{R})$ :

$$C^h(S^1, \mathbb{R}) := \{f(z) \in \mathcal{O}(\mathbb{C}^*) ; f(z) \in \mathbb{R} \text{ if } |z| = 1\}.$$

Note that we have  $\mathcal{O}(\mathbb{C}^*) = C^h(S^1, \mathbb{R}) \oplus \mathbf{i}C^h(S^1, \mathbb{R})$ . The involution  $\kappa$  on  $\mathcal{O}(\mathbb{C}^*)$  corresponding to the real form  $C^h(S^1, \mathbb{R})$  is given by

$$\kappa(f)(z) = \overline{f(\gamma(z))}, \quad f(z) \in \mathcal{O}(\mathbb{C}^*),$$

where  $\gamma(z) = 1/\bar{z}$  and the  $\overline{\phantom{x}}$  in the right-hand side is the complex conjugate. We also have  $\mathcal{H} \cong \mathcal{H}_{\mathbb{R}} \oplus \mathbf{i}\mathcal{H}_{\mathbb{R}}$ . This real form  $\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}$  defines an involution  $\kappa_{\mathcal{H}}: \mathcal{H} \rightarrow \mathcal{H}$  such that  $\kappa_{\mathcal{H}}(\alpha + \mathbf{i}\beta) = \alpha - \mathbf{i}\beta$  for  $\alpha, \beta \in \mathcal{H}_{\mathbb{R}}$ . This satisfies

$$(10) \quad \begin{aligned} \kappa_{\mathcal{H}}(fs) &= \kappa(f)\kappa_{\mathcal{H}}(s), \\ \kappa_{\mathcal{H}}\widehat{\nabla}_{z\partial_z} &= -\widehat{\nabla}_{z\partial_z}\kappa_{\mathcal{H}}, \\ \kappa((s_1, s_2)_{\mathcal{H}}) &= (\kappa_{\mathcal{H}}(s_1), \kappa_{\mathcal{H}}(s_2))_{\mathcal{H}}. \end{aligned}$$

Note that  $\kappa_{\mathcal{H}}$  matches with the real involution on  $R_{(\tau,z)}$  over the equator  $\{|z| = 1\}$ . Similarly, we have  $\mathcal{V} = \mathcal{V}_{\mathbb{R}} \oplus \mathbf{i}\mathcal{V}_{\mathbb{R}}$ ; we denote by  $\kappa_{\mathcal{V}}: \mathcal{V} \rightarrow \mathcal{V}$  the involution defined by the real structure  $\mathcal{V}_{\mathbb{R}}$ .

**Remark 2.6.** In the context of the smooth Grassmannian, it is more natural to work over  $C^\infty(S^1, \mathbb{C})$  instead of  $\mathcal{O}(\mathbb{C}^*)$ , where  $S^1 = \{|z| = 1\}$ . We put

$$\widetilde{\mathcal{H}} := \mathcal{H} \otimes_{\mathcal{O}(\mathbb{C}^*)} C^\infty(S^1, \mathbb{C}), \quad \widetilde{\mathbb{F}}_\tau := \mathbb{F}_\tau \otimes_{\mathcal{O}(\mathbb{C})} \mathcal{O}(\mathbb{D}_0),$$

where  $\mathcal{O}(\mathbb{D}_0)$  is a subspace of  $C^\infty(S^1, \mathbb{C})$  consisting of functions which are the boundary values of holomorphic functions on the interior of the disc  $\mathbb{D}_0 = \{z \in \mathbb{C} ; |z| \leq 1\}$ . The involution  $\kappa_{\mathcal{H}}: \widetilde{\mathcal{H}} \rightarrow \widetilde{\mathcal{H}}$  and the real form  $\widetilde{\mathcal{H}}_{\mathbb{R}}$  is defined similarly and the same properties hold. Conversely, using the flat connection  $\widehat{\nabla}_{z\partial_z}$  in the  $z$ -direction, one can recover  $\mathbb{F}_\tau$  from  $\widetilde{\mathbb{F}}_\tau$  since flat sections of  $\widehat{\nabla}_{z\partial_z}$  determine an extension of the bundle on  $\mathbb{D}_0$  to  $\mathbb{C}$ .

**2.3. Pure and polarized  $\frac{\infty}{2}$ VHS.** Following Hertling [21], we define an extension  $\widehat{K}$  of  $\mathcal{R}^{(0)}$  across  $z = \infty$ . The properties “pure and polarized” for  $\mathcal{F}$  are defined in terms of this extension.

**Definition 2.7** (Extension of  $\mathcal{R}^{(0)}$  across  $z = \infty$ ). Let  $\gamma: \mathcal{M} \times \mathbb{P}^1 \rightarrow \mathcal{M} \times \mathbb{P}^1$  be the map defined by  $\gamma(\tau, z) = (\tau, 1/\bar{z})$ . Let  $\overline{\mathcal{M}}$  denote the complex conjugate of  $\mathcal{M}$ , i.e.  $\overline{\mathcal{M}}$  is the same as  $\mathcal{M}$  as a real-analytic manifold but holomorphic functions on  $\overline{\mathcal{M}}$  are anti-holomorphic functions on  $\mathcal{M}$ . The pull-back  $\gamma^*\mathcal{R}^{(0)}$  of  $\mathcal{R}^{(0)}$  has the structure of an  $\mathcal{O}_{\mathcal{M} \times (\mathbb{P}^1 \setminus \{0\})}$ -module. Thus its complex conjugate  $\overline{\gamma^*\mathcal{R}^{(0)}}$  has the structure of an  $\mathcal{O}_{\overline{\mathcal{M}} \times (\mathbb{P}^1 \setminus \{0\})}$ -module. Regarding  $\mathcal{R}^{(0)}$  and  $\overline{\gamma^*\mathcal{R}^{(0)}}$  as real-analytic vector bundles over  $\mathcal{M} \times \mathbb{C}$  and  $\mathcal{M} \times (\mathbb{P}^1 \setminus \{0\})$ , we glue them along  $\mathcal{M} \times \mathbb{C}^*$  by the fiberwise map

$$(11) \quad \mathcal{R}_{(\tau, z)}^{(0)} \xrightarrow{\kappa} \overline{\mathcal{R}_{(\tau, z)}^{(0)}} \xrightarrow{P(\gamma(z), z)} \overline{\mathcal{R}_{(\tau, \gamma(z))}^{(0)}} = \overline{\gamma^*\mathcal{R}_{(\tau, z)}^{(0)}}, \quad z \in \mathbb{C}^*.$$

Here the first map  $\kappa$  is the real involution on  $\mathcal{R}_{(\tau, z)}^{(0)}$  with respect to the real form  $R_{\mathbb{R}, (\tau, z)}$  and the second map  $P(\gamma(z), z)$  is the parallel translation for the flat connection  $\widehat{\nabla}$  along the path  $[0, 1] \ni t \mapsto (1-t)z + t\gamma(z)$ . Define  $\widehat{K} \rightarrow \mathcal{M} \times \mathbb{P}^1$  to be the real-analytic complex vector bundle obtained by gluing  $\mathcal{R}^{(0)}$  and  $\overline{\gamma^*\mathcal{R}^{(0)}}$  in this way. Notice that  $\widehat{K}|_{\tau \times \mathbb{P}^1}$  has the structure of a holomorphic vector bundle since the gluing map (11) preserves the holomorphic structure in the  $\mathbb{P}^1$ -direction.  $\square$

**Definition 2.8.** A graded  $\frac{\infty}{2}$ VHS  $\mathcal{F}$  with a real structure is called *pure* at  $\tau \in \mathcal{M}$  if  $\widehat{K}|_{\{\tau\} \times \mathbb{P}^1}$  is trivial as a holomorphic vector bundle on  $\mathbb{P}^1$ .  $\square$

A pure graded  $\frac{\infty}{2}$ VHS with a real structure here corresponds to the (trTERP) structure in [21]. Here we follow the terminology in [22].

We rephrase the purity in terms of the moving subspace realization  $\{\mathbb{F}_\tau \subset \mathcal{H}\}$ . When we identify  $\mathcal{H}$  with the space of global sections of  $\widehat{K}|_{\{\tau\} \times \mathbb{C}^*} = \mathcal{R}|_{\{\tau\} \times \mathbb{C}^*}$ , it is easy to see that the involution  $\kappa_{\mathcal{H}}: \mathcal{H} \rightarrow \mathcal{H}$  is induced by the gluing map (11). Then  $\mathbb{F}_\tau$  is identified with the space of holomorphic sections of  $\widehat{K}|_{\{\tau\} \times \mathbb{C}^*}$  which can extend to  $\{\tau\} \times \mathbb{C}$ ;  $\kappa_{\mathcal{H}}(\mathbb{F}_\tau)$  is identified with the space of holomorphic sections of  $\widehat{K}|_{\{\tau\} \times \mathbb{C}^*}$  which can extend to  $\{\tau\} \times (\mathbb{P}^1 \setminus \{0\})$ . Similarly,  $\widetilde{\mathbb{F}}_\tau$  (resp.  $\kappa_{\mathcal{H}}(\widetilde{\mathbb{F}}_\tau)$ ) is identified with the space of smooth sections of  $\widehat{K}|_{\{\tau\} \times S^1}$  which can extend to holomorphic sections on  $\mathbb{D}_0$  (resp.  $\mathbb{D}_\infty$ ), where  $\mathbb{D}_0 = \{z \in \mathbb{C} ; |z| \leq 1\}$ ,  $\mathbb{D}_\infty = \{z \in \mathbb{C} \cup \{\infty\} = \mathbb{P}^1 ; |z| \geq 1\}$  and  $\widetilde{\mathbb{F}}_\tau$  is the space in Remark 2.6.

**Proposition 2.9.** *A graded  $\frac{\infty}{2}$ VHS  $\mathcal{F}$  with a real structure is pure at  $\tau \in \mathcal{M}$  if and only if one of the following natural maps is an isomorphism:*

$$(12) \quad \mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) \longrightarrow \mathbb{F}_\tau / z\mathbb{F}_\tau,$$

$$(13) \quad (\mathbb{F}_\tau \cap \mathcal{H}_{\mathbb{R}}) \otimes \mathbb{C} \longrightarrow \mathbb{F}_\tau / z\mathbb{F}_\tau,$$

$$(14) \quad \mathbb{F}_\tau \oplus z^{-1}\kappa_{\mathcal{H}}(\mathbb{F}_\tau) \longrightarrow \mathcal{H}.$$

*This holds also true when  $\mathbb{F}_\tau$ ,  $\mathcal{H}$ ,  $\mathcal{H}_{\mathbb{R}}$  are replaced with  $\widetilde{\mathbb{F}}_\tau$ ,  $\widetilde{\mathcal{H}}$ ,  $\widetilde{\mathcal{H}}_{\mathbb{R}}$  in Remark 2.6. When  $\mathcal{F}$  is pure at some  $\tau$ ,  $\mathcal{H}_{\mathbb{R}}$  is a free module over  $C^h(S^1, \mathbb{R})$ .*



*Proof.* Under the identifications we explained above,  $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$  is identified with the space of global sections of  $\widehat{K}|_{\{\tau\} \times \mathbb{P}^1}$  and the natural map  $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) \rightarrow \mathbb{F}_\tau/z\mathbb{F}_\tau$  corresponds to the restriction to  $z = 0$  (note that  $\mathbb{F}_\tau/z\mathbb{F}_\tau \cong \widehat{K}_{(\tau,0)}$ ). Therefore (12) is an isomorphism if and only if  $K|_{\{\tau\} \times \mathbb{P}^1}$  is trivial.  $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$  is invariant under  $\kappa_{\mathcal{H}}$  and its real form is given by  $\mathbb{F}_\tau \cap \mathcal{H}_{\mathbb{R}}$ . Therefore, we have  $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) \cong (\mathbb{F}_\tau \cap \mathcal{H}_{\mathbb{R}}) \otimes \mathbb{C}$ . Thus (12) is an isomorphism if and only if so is (13). Similarly, we can see that (14) is an isomorphism if  $\widehat{K}|_{\{\tau\} \times \mathbb{P}^1}$  is trivial. Conversely, we show that (12) is an isomorphism if so is (14). The injectivity of the map  $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) \rightarrow \mathbb{F}_\tau/z\mathbb{F}_\tau$  is easy to check. Take  $v \in \mathbb{F}_\tau$ . By assumption,  $z^{-1}v = v_1 + v_2$  for some  $v_1 \in \mathbb{F}_\tau$  and  $v_2 \in z^{-1}\kappa_{\mathcal{H}}(\mathbb{F}_\tau)$ . Thus  $v - zv_1 = zv_2 \in \mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$  and the image of this element in  $\mathbb{F}_\tau/z\mathbb{F}_\tau$  is  $[v]$ . The discussion on the spaces  $\widetilde{\mathbb{F}}_\tau$ ,  $\widetilde{\mathcal{H}}$  and  $\widetilde{\mathcal{H}}_{\mathbb{R}}$  are similar.

The last statement: Since  $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) \cong (\mathbb{F}_\tau \cap \mathcal{H}_{\mathbb{R}}) \otimes \mathbb{C}$ , we can take a global basis of the trivial bundle  $\widehat{K}|_{\{\tau\} \times \mathbb{P}^1}$  from  $\mathbb{F}_\tau \cap \mathcal{H}_{\mathbb{R}}$ . The module  $\mathcal{H}_{\mathbb{R}}$  is freely generated by such a basis over  $C^h(S^1, \mathbb{R})$ .  $\square$

**Definition 2.10.** A graded  $\frac{\infty}{2}$ VHS  $\mathcal{F}$  with a real structure is called *polarized* at  $\tau \in \mathcal{M}$  if the Hermitian pairing  $h$  on  $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) \cong \Gamma(\mathbb{P}^1, \widehat{K}|_{\{\tau\} \times \mathbb{P}^1})$  defined by

$$h: s_1 \times s_2 \longmapsto (\kappa_{\mathcal{H}}(s_1), s_2)_{\mathcal{H}}$$

is positive definite. Note that this pairing takes values in  $\mathbb{C}$  since  $(\mathbb{F}_\tau, \mathbb{F}_\tau)_{\mathcal{H}} \subset \mathcal{O}(\mathbb{C})$  and  $(\kappa_{\mathcal{H}}(\mathbb{F}_\tau), \kappa_{\mathcal{H}}(\mathbb{F}_\tau))_{\mathcal{H}} \subset \mathcal{O}(\mathbb{P}^1 \setminus \{0\})$  by (10). It is easy to show that a polarized  $\frac{\infty}{2}$ VHS is necessarily pure at the same point.  $\square$

**Remark 2.11.** In order to obtain a basis of  $\widetilde{\mathbb{F}}_\tau \cap \kappa_{\mathcal{H}}(\widetilde{\mathbb{F}}_\tau)$  or  $\widetilde{\mathbb{F}}_\tau \cap \widetilde{\mathcal{H}}_{\mathbb{R}}$ , we can make use of Birkhoff or Iwasawa factorization. Take an  $\mathcal{O}(\mathbb{D}_0)$ -basis  $s_1, \dots, s_N$  of  $\widetilde{\mathbb{F}}_\tau$ . Define an element  $A(z) = (A_{ij}(z))$  of the loop group  $LGL_N(\mathbb{C})$  by

$$[\kappa_{\mathcal{H}}(s_1), \dots, \kappa_{\mathcal{H}}(s_N)] = [s_1, \dots, s_N]A(z), \quad i.e. \quad \kappa_{\mathcal{H}}(s_i) = \sum_j s_j A_{ji}(z).$$

If  $A(z)$  admits the Birkhoff factorization  $A(z) = B(z)C(z)$ , where  $B(z)$  and  $C(z)$  are holomorphic maps  $B(z): \mathbb{D}_0 \rightarrow GL_N(\mathbb{C})$ ,  $C(z): \mathbb{D}_\infty \rightarrow GL_N(\mathbb{C})$  such that  $B(0) = \mathbf{1}$ , then we obtain a  $\mathbb{C}$ -basis of  $\widetilde{\mathbb{F}}_\tau \cap \kappa_{\mathcal{H}}(\widetilde{\mathbb{F}}_\tau)$  as

$$(15) \quad [\kappa_{\mathcal{H}}(s_1), \dots, \kappa_{\mathcal{H}}(s_N)]C(z)^{-1} = [s_1, \dots, s_N]B(z).$$

Here,  $\mathcal{F}$  is pure at  $\tau \in \mathcal{M}$  if and only if  $A(z)$  admits the Birkhoff factorization, *i.e.*  $A(z)$  is in the “big cell” of the loop group. In particular, the purity is an open condition for  $\tau \in \mathcal{M}$ . On the other hand, the Iwasawa-type factorization appears as follows. Assume that we have a basis  $e_1, \dots, e_N$  of  $\widetilde{\mathcal{H}}_{\mathbb{R}}$  over  $C^\infty(S^1, \mathbb{R})$  such that  $(e_i, e_j)_{\widetilde{\mathcal{H}}} = \delta_{ij}$  and a basis  $s_1, \dots, s_N$  of  $\widetilde{\mathbb{F}}_\tau$  over  $\mathcal{O}(\mathbb{D}_0)$  such that  $(s_i, s_j)_{\widetilde{\mathcal{H}}} = \delta_{ij}$ . Define a matrix  $J(z)$  by

$$[s_1, \dots, s_N] = [e_1, \dots, e_N]J(z).$$

This  $J(z)$  lies in the *twisted loop group*  $LGL_N(\mathbb{C})_{\text{tw}}$ :

$$LGL_N(\mathbb{C})_{\text{tw}} := \{J: S^1 \rightarrow GL_N(\mathbb{C}) ; J(-z)^T J(z) = \mathbf{1}\}.$$

If  $J(z)$  admits an Iwasawa-type factorization  $J(z) = U(z)B(z)$ , where  $U: S^1 \rightarrow GL_N(\mathbb{R})$  with  $U(-z)^T U(z) = \mathbf{1}$  and  $B: \mathbb{D}_0 \rightarrow GL_N(\mathbb{C})$  with  $B(-z)^T B(z) = \mathbf{1}$ , then we obtain an  $\mathbb{R}$ -basis of  $\tilde{\mathbb{F}}_\tau \cap \tilde{\mathcal{H}}_\mathbb{R}$  as

$$[s_1, \dots, s_N]B(z)^{-1} = [e_1, \dots, e_N]U(z)$$

which is orthonormal with respect to  $(\cdot, \cdot)_{\tilde{\mathcal{H}}}$ . In this case, the pairing  $(\cdot, \cdot)_{\tilde{\mathcal{H}}}$  restricted to  $\tilde{\mathbb{F}}_\tau \cap \tilde{\mathcal{H}}_\mathbb{R}$  is an  $\mathbb{R}$ -valued *positive definite* symmetric form. The map  $\tau \mapsto J(z)$  gives rise to the semi-infinite period map in Section 2.2:

$$\mathcal{M} \ni \tau \longmapsto [J(z)] \in LGL_N(\mathbb{C})_{\text{tw}} / LGL_N^+(\mathbb{C})_{\text{tw}}.$$

Here,  $\mathcal{F}$  is pure at  $\tau$  and  $(\tilde{\mathbb{F}}_\tau \cap \tilde{\mathcal{H}}_\mathbb{R}, (\cdot, \cdot)_{\tilde{\mathcal{H}}})$  is positive definite if and only if the image of this map lies in the  $LGL_N(\mathbb{R})_{\text{tw}}$ -orbit of  $[\mathbf{1}]$ . This orbit is open, but not dense. We owe the Lie group theoretic viewpoint here to Guest [19, 20].

**Remark 2.12.** In addition to the purity and the polarization, Hertling-Sevenheck [22] and Katzarkov-Kontsevich-Pantev [26] considered the compatibility of a real (or rational) structure and the Stokes structure.

**2.4. Cecotti-Vafa structure.** We describe the Cecotti-Vafa structure ( $tt^*$ -geometry) associated to a pure graded  $\frac{\infty}{2}$ VHS with a real structure.

Define a complex vector bundle  $K \rightarrow \mathcal{M}$  by  $K := \hat{K}|_{\mathcal{M} \times \{0\}}$ . This is the real analytic vector bundle underlying  $\mathcal{F}/z\mathcal{F} \cong \mathcal{R}^{(0)}|_{\mathcal{M} \times \{0\}}$ . Let  $\mathcal{A}_\mathcal{M}^p$  be the sheaf of complex-valued  $C^\infty$   $p$ -forms on  $\mathcal{M}$  and  $\mathcal{A}_\mathcal{M}^1 = \mathcal{A}_\mathcal{M}^{1,0} \oplus \mathcal{A}_\mathcal{M}^{0,1}$  be the type decomposition.

**Proposition 2.13** ([21, Theorem 2.19]). *Assume that a graded  $\frac{\infty}{2}$ VHS  $\mathcal{F}$  with a real structure is pure over  $\mathcal{M}$ . Then the vector bundle  $K$  is equipped with a Cecotti-Vafa structure  $(\kappa, g, C, \tilde{C}, D, \mathcal{Q}, \mathcal{U}, \overline{\mathcal{U}})$ . This is given by the data (see (16), (17), (18), (19)):*

- A complex-antilinear involution  $\kappa: K_\tau \rightarrow K_\tau$ ;
- A non-degenerate, symmetric,  $\mathbb{C}$ -bilinear metric  $g: K_\tau \times K_\tau \rightarrow \mathbb{C}$  which is real with respect to  $\kappa$ , i.e.  $g(\kappa u_1, \kappa u_2) = \overline{g(u_1, u_2)}$ ;
- Endomorphisms  $C \in \text{End}(K) \otimes \mathcal{A}_\mathcal{M}^{1,0}$ ,  $\tilde{C} \in \text{End}(K) \otimes \mathcal{A}_\mathcal{M}^{0,1}$  such that  $\tilde{C}_i = \kappa C_i \kappa$ ;
- A connection  $D: K \rightarrow K \otimes \mathcal{A}_\mathcal{M}^1$  real with respect to  $\kappa$ , i.e.  $D_i = \kappa D_i \kappa$ ;
- Endomorphisms  $\mathcal{Q}, \mathcal{U}, \overline{\mathcal{U}} \in \text{End}(K)$  such that  $\mathcal{U} = C_E$ ,  $\overline{\mathcal{U}} = \kappa \mathcal{U} \kappa = \tilde{C}_{\overline{E}}$  and  $\mathcal{Q} \kappa = -\kappa \mathcal{Q}$

satisfying the integrability conditions

$$\begin{aligned} [D_i, D_j] &= 0, & D_i C_j - D_j C_i &= 0, & [C_i, C_j] &= 0, \\ [D_{\overline{i}}, D_{\overline{j}}] &= 0, & D_{\overline{i}} \tilde{C}_{\overline{j}} - D_{\overline{j}} \tilde{C}_{\overline{i}} &= 0, & [\tilde{C}_{\overline{i}}, \tilde{C}_{\overline{j}}] &= 0, \\ D_i \tilde{C}_{\overline{j}} &= 0, & D_{\overline{i}} C_j &= 0, & [D_i, D_{\overline{j}}] + [C_i, \tilde{C}_{\overline{j}}] &= 0, \\ D_i \overline{\mathcal{U}} &= 0, & D_i \mathcal{Q} - [\overline{\mathcal{U}}, C_i] &= 0, & D_i \mathcal{U} - C_i + [\mathcal{Q}, C_i] &= 0, & [\mathcal{U}, C_i] &= 0, \\ D_{\overline{i}} \mathcal{U} &= 0, & D_{\overline{i}} \mathcal{Q} + [\mathcal{U}, \tilde{C}_{\overline{i}}] &= 0, & D_{\overline{i}} \overline{\mathcal{U}} - \tilde{C}_{\overline{i}} - [\mathcal{Q}, \tilde{C}_{\overline{i}}] &= 0, & [\overline{\mathcal{U}}, \tilde{C}_{\overline{i}}] &= 0, \end{aligned}$$

and the compatibility with the metric

$$\begin{aligned}\partial_i g(u_1, u_2) &= g(D_i u_1, u_2) + g(u_1, D_i u_2), \\ \partial_{\bar{i}} g(u_1, u_2) &= g(D_{\bar{i}} u_1, u_2) + g(u_1, D_{\bar{i}} u_2), \\ g(C_i u_1, u_2) &= g(u_1, C_i u_2), \quad g(\tilde{C}_{\bar{i}} u_1, u_2) = g(u_1, \tilde{C}_{\bar{i}} u_2), \\ g(\mathcal{U} u_1, u_2) &= g(u_1, \mathcal{U} u_2), \quad g(\overline{\mathcal{U}} u_1, u_2) = g(u_1, \overline{\mathcal{U}} u_2), \\ g(\mathcal{Q} u_1, u_2) + g(u_1, \mathcal{Q} u_2) &= 0.\end{aligned}$$

Here we chose a local complex co-ordinate system  $\{t^i\}$  on  $\mathcal{M}$  and used the notation  $D_i = D_{\partial/\partial t^i}$ ,  $D_{\bar{i}} = D_{\partial/\partial \bar{t}^i}$ , etc. The Hermitian metric  $h$  in Definition 2.10 is related to  $g$  by

$$h(u_1, u_2) = g(\kappa(u_1), u_2).$$

A concrete example of the Cecotti-Vafa structure will be given in Section 4. We explain the construction of the above data from the  $\frac{\infty}{2}$ VHS  $\mathcal{F}$ . Because  $\mathcal{F}$  is pure, we have a canonical identification

$$\Phi_\tau: K_\tau \cong \Gamma(\mathbb{P}^1, \widehat{K}|_{\{\tau\} \times \mathbb{P}^1}) \cong \mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau).$$

The involution  $\kappa_{\mathcal{H}}$  and the pairing  $(\cdot, \cdot)_{\mathcal{H}}$  restricted to  $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$  induce an involution  $\kappa$  and a  $\mathbb{C}$ -bilinear pairing  $g$  on  $K_\tau$ :

$$(16) \quad \Phi_\tau(\kappa(u)) := \kappa_{\mathcal{H}}(\Phi_\tau(u)),$$

$$(17) \quad g(u_1, u_2) := (\Phi_\tau(u_1), \Phi_\tau(u_2))_{\mathcal{H}} \in \mathbb{C}$$

satisfying

$$g(\kappa u_1, \kappa u_2) = \overline{g(u_1, u_2)}, \quad g(u_1, u_2) = g(u_2, u_1).$$

Note that the subspace  $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$  depends on the parameter  $\tau$  real analytically. A  $C^\infty$ -version of the Griffiths transversality gives

$$\begin{aligned}X^{(1,0)} \mathbb{F}_\tau &\subset z^{-1} \mathbb{F}_\tau, & X^{(0,1)} \mathbb{F}_\tau &\subset \mathbb{F}_\tau, \\ X^{(1,0)} \kappa_{\mathcal{H}}(\mathbb{F}_\tau) &\subset \kappa_{\mathcal{H}}(\mathbb{F}_\tau), & X^{(0,1)} \kappa_{\mathcal{H}}(\mathbb{F}_\tau) &\subset z \kappa_{\mathcal{H}}(\mathbb{F}_\tau),\end{aligned}$$

where  $X^{(1,0)} \in T_\tau^{1,0} \mathcal{M}$  and  $X^{(0,1)} \in T_\tau^{0,1} \mathcal{M}$ . For  $X^{(1,0)} \in T_\tau^{1,0} \mathcal{M}$ , we have

$$X^{(1,0)}(\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)) \subset z^{-1} \mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) = z^{-1}(\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)) \oplus (\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)).$$

Similarly for  $X^{(0,1)} \in T_\tau^{(0,1)} \mathcal{M}$ , we have

$$X^{(0,1)}(\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)) \subset (\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)) \oplus z(\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)).$$

Hence we can define endomorphisms  $C: K \rightarrow K \otimes \mathcal{A}^{1,0}$ ,  $\tilde{C}: K \rightarrow K \otimes \mathcal{A}^{0,1}$ , and a connection  $D: K \rightarrow K \otimes \mathcal{A}^1$  by

$$(18) \quad X \Phi_\tau(u_\tau) = z^{-1} \Phi_\tau(C_X(u_\tau)) + \Phi_\tau(D_X(u_\tau)) + z \Phi_\tau(\tilde{C}_X(u_\tau))$$

for a section  $u_\tau$  of  $K$ . By applying  $\kappa_{\mathcal{H}}$  on the both hand sides,

$$\overline{X} \Phi_\tau(\kappa u_\tau) = z^{-1} \Phi_\tau(\kappa \tilde{C}_X(u_\tau)) + \Phi_\tau(\kappa D_X(u_\tau)) + z \Phi_\tau(\kappa C_X(u_\tau)).$$

Therefore, we must have

$$C_{\overline{X}} \kappa = \kappa \tilde{C}_X, \quad \kappa D_X = D_{\overline{X}} \kappa, \quad X \in T\mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}.$$

Similarly, we can define endomorphisms  $\mathcal{U}, \overline{\mathcal{U}}, \mathcal{Q}: K \rightarrow K$  by

$$(19) \quad \widehat{\nabla}_{z\partial_z} \Phi_\tau(u_\tau) = -z^{-1} \Phi_\tau(\mathcal{U}(u_\tau)) + \Phi_\tau(\mathcal{Q}(u_\tau)) + z \Phi_\tau(\overline{\mathcal{U}}(u_\tau)).$$

Because  $\widehat{\nabla}_{z\partial_z}$  is purely imaginary (10), we have

$$\kappa \mathcal{Q} = -\mathcal{Q} \kappa, \quad \overline{\mathcal{U}} = \kappa \mathcal{U} \kappa.$$

By  $(\widehat{\nabla}_{z\partial_z} + E)\mathbb{F}_\tau \subset \mathbb{F}_\tau$  in Proposition 2.5, we find

$$\mathcal{U} = C_E, \quad \overline{\mathcal{U}} = \tilde{C}_{\overline{E}}.$$

We have a canonical isomorphism

$$\pi^* K \cong \widehat{K}, \quad \text{where } \pi: \mathcal{M} \times \mathbb{P}^1 \rightarrow \mathcal{M}.$$

Let  $C^{\infty h}(\pi^* K)$  be the sheaf of  $C^\infty$  sections of  $\pi^* K \cong \widehat{K}$  which are holomorphic on each fiber  $\{\tau\} \times \mathbb{P}^1$ . Under the isomorphism above, the flat connection  $\widehat{\nabla}$  on  $\mathcal{R}^{(0)} = \widehat{K}|_{\mathcal{M} \times \mathbb{C}}$  can be written in the form:

$$(20) \quad \begin{aligned} \widehat{\nabla}: C^{\infty h}(\pi^* K) &\longrightarrow C^{\infty h}(\pi^* K) \otimes \left( z^{-1} \mathcal{A}_{\mathcal{M}}^{1,0} \oplus \mathcal{A}_{\mathcal{M}}^1 \oplus z \mathcal{A}_{\mathcal{M}}^{0,1} \right. \\ &\quad \left. \oplus (z^{-1} \mathcal{A}_{\mathcal{M}}^0 \oplus \mathcal{A}_{\mathcal{M}}^0 \oplus z \mathcal{A}_{\mathcal{M}}^0) \frac{dz}{z} \right) \\ \widehat{\nabla} &= z^{-1} C + D + z \tilde{C} + (z \partial_z - z^{-1} \mathcal{U} + \mathcal{Q} + z \overline{\mathcal{U}}) \otimes \frac{dz}{z}. \end{aligned}$$

Under the same isomorphism, the pairing  $(\cdot, \cdot)_{\mathcal{R}^{(0)}}$  on  $\mathcal{R}^{(0)} = \widehat{K}|_{\mathcal{M} \times \mathbb{C}}$  can be written as

$$\begin{aligned} C^{\infty h}((-)^*(\pi^* K)) \otimes C^{\infty h}(\pi^* K) &\rightarrow C^{\infty h}(\mathcal{M} \times \mathbb{P}^1) \\ s_1(\tau, -z) \otimes s_2(\tau, z) &\longmapsto g(s_1(\tau, -z), s_2(\tau, z)). \end{aligned}$$

Unpacking the flatness of  $\widehat{\nabla}$  and  $\widehat{\nabla}$ -flatness of the pairing in terms of  $C, \tilde{C}, D, \mathcal{U}, \mathcal{Q}$  and  $g$ , we arrive at the equations in Proposition 2.13.

**Remark 2.14.** (i) The  $(0, 1)$ -part  $\widehat{\nabla}_{\bar{\tau}} = D_{\bar{\tau}} + z \tilde{C}_{\bar{\tau}}$  of the flat connection (20) gives the holomorphic structure on  $\widehat{K}|_{\mathcal{M} \times \{z\}}$  which corresponds to the holomorphic structure on  $\mathcal{R}^{(0)}$ . In particular,  $D$  is identified with the canonical connection associated to the Hermitian metric  $h$  on the holomorphic vector bundle  $\mathcal{F}/z\mathcal{F}$ . Similarly, the  $(1, 0)$ -part  $D_i + z^{-1} C_i$  gives an anti-holomorphic structure on  $\widehat{K}|_{\mathcal{M} \times \{z\}}$  which corresponds to the anti-holomorphic structure on  $\overline{\gamma^* \mathcal{R}^{(0)}}$ .

(ii) Among the data of the Cecotti-Vafa structure, one can define the data  $(C, D_E + \mathcal{Q}, \mathcal{U}, g)$  without choosing a real structure. In fact,  $C_X$  is given by the map  $\mathcal{F}/z\mathcal{F} \ni [s] \mapsto [z \nabla_X s] \in \mathcal{F}/z\mathcal{F}$ ,  $D_E + \mathcal{Q}$  is given by the map  $\mathcal{F}/z\mathcal{F} \ni [s] \mapsto [\frac{1}{2}(\text{Gr} - n)s] \in \mathcal{F}/z\mathcal{F}$ ,  $\mathcal{U} = C_E$ , and  $g$  is given by  $g([s_1], [s_2]) = (s_1, s_2)_{\mathcal{F}|_{z=0}}$  for  $s_i \in \mathcal{F}$ . In the case of quantum cohomology,  $C_i$  is the quantum multiplication  $\phi_i \circ$  by some  $\phi_i \in H_{\text{orb}}^*(\mathcal{X})$  (see (23), (25)) and  $g$  is the Poincaré pairing.

**Remark 2.15.** A Frobenius manifold structure [15] on  $\mathcal{M}$  arises from a miniversal  $\frac{\infty}{2}$ VHS (in the sense of [12, Definition 2.8]) without a real structure. To obtain a

Frobenius manifold structure, we need a choice of an opposite subspace  $\mathcal{H}_- \subset \mathcal{H}$ : a sub free  $\mathcal{O}(\mathbb{P}^1 \setminus \{0\})$ -module  $\mathcal{H}_-$  of  $\mathcal{H}$  satisfying

$$\mathcal{H} = \mathbb{F}_\tau \oplus \mathcal{H}_-, \quad \widehat{\nabla}_{z\partial_z} \mathcal{H}_- \subset \mathcal{H}_-.$$

The choice of  $\mathcal{H}_-$  corresponds to giving a logarithmic extension of the flat vector bundle  $(H, \widehat{\nabla}_{z\partial_z})$  at  $z = \infty$ . A graded  $\frac{\infty}{2}$ VHS with the choice of an opposite subspace corresponds to the (trTLEP)-structure in Hertling [21]. See [3, 21, 12] for the construction of Frobenius manifolds from this viewpoint. In the  $tt^*$ -geometry, the complex conjugate  $\kappa_{\mathcal{H}}(\mathbb{F}_\tau)$  of the Hodge structure  $\mathbb{F}_\tau$  plays the role of the opposite subspace (see (14)). When a miniversal  $\frac{\infty}{2}$ VHS is equipped with both a real structure and an opposite subspace, under certain conditions,  $\mathcal{M}$  has a CDV (Cecotti-Dubrovin-Vafa) structure, which dominates both Frobenius manifold structure and Cecotti-Vafa structure on  $T\mathcal{M}$ . See [21, Theorem 5.15] for more details.

### 3. REAL STRUCTURES ON THE QUANTUM COHOMOLOGY

In this section, we give a review of orbifold quantum cohomology and introduce a real structure on it. Some of the basic materials here have overlaps with the companion paper [24] and we refer the reader to it for the proofs.

**3.1. Orbifold quantum cohomology.** Quantum cohomology for orbifolds have been developed by Chen-Ruan [10] for symplectic orbifolds and Abramovich-Graber-Vistoli [1] for smooth Deligne-Mumford stacks. Real structures make sense for both (symplectic and algebraic) categories, but we will work in the algebraic category. For example, we need the Lefschetz decomposition in the proof of Theorem 3.9.

Let  $\mathcal{X}$  be a proper smooth Deligne-Mumford stack over  $\mathbb{C}$ . Let  $I\mathcal{X}$  be the *inertia stack* of  $\mathcal{X}$ , which is defined to be the fiber product  $\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$  of the two diagonal morphisms  $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ . A point of  $I\mathcal{X}$  is given by a pair  $(x, g)$  of a point  $x \in \mathcal{X}$  and  $g \in \text{Aut}(x)$ . Here  $g$  is called the *stabilizer* at  $(x, g) \in I\mathcal{X}$ . The inertia stack is decomposed into connected components:

$$I\mathcal{X} = \bigsqcup_{v \in \mathbb{T}} \mathcal{X}_v = \mathcal{X}_0 \cup \bigsqcup_{v \in \mathbb{T}'} \mathcal{X}_v, \quad \mathcal{X}_0 = \mathcal{X}.$$

Here  $\mathbb{T}$  is the index set of connected components,  $0 \in \mathbb{T}$  corresponds to the distinguished component with the trivial stabilizer and  $\mathbb{T}' = \mathbb{T} \setminus \{0\}$ . For each connected component  $\mathcal{X}_v$  of  $I\mathcal{X}$ , we associate a rational number  $\iota_v$  called *age*. For  $(x, g) \in \mathcal{X}_v \subset I\mathcal{X}$ , let  $0 \leq f_1, \dots, f_n < 1$  ( $n = \dim_{\mathbb{C}} \mathcal{X}$ ) be rational numbers such that the stabilizer  $g$  acts on the tangent space  $T_x \mathcal{X}$  with eigenvalues  $\exp(2\pi \mathbf{i} f_1), \dots, \exp(2\pi \mathbf{i} f_n)$  (with multiplicities). Then we set

$$\iota_v := f_1 + \dots + f_n.$$

The (*even parity*) *orbifold cohomology group*  $H_{\text{orb}}^*(\mathcal{X})$  is defined to be

$$H_{\text{orb}}^k(\mathcal{X}) = \bigoplus_{v \in \mathbb{T}: k - 2\iota_v \in 2\mathbb{Z}} H^{k-2\iota_v}(\mathcal{X}_v, \mathbb{C}).$$

The degree  $k$  of the orbifold cohomology can be a fractional number in general. Each factor  $H^*(\mathcal{X}_v, \mathbb{C})$  in the right-hand side denotes the cohomology group of  $\mathcal{X}_v$  as a

topological space. We define an involution  $\text{inv}: I\mathcal{X} \rightarrow I\mathcal{X}$  by  $\text{inv}(x, g) = (x, g^{-1})$  and the *orbifold Poincaré pairing* by

$$(\alpha, \beta)_{\text{orb}} := \int_{I\mathcal{X}} \alpha \cup \text{inv}^*(\beta) = \sum_{v \in \mathbb{T}} \int_{\mathcal{X}_v} \alpha_v \cup \beta_{\text{inv}(v)}.$$

where  $\alpha_v, \beta_v$  are the  $\mathcal{X}_v$ -components of  $\alpha, \beta$  and  $\text{inv}: \mathbb{T} \rightarrow \mathbb{T}$  denotes the induced involution on  $\mathbb{T}$ . This is a symmetric non-degenerate pairing of degree  $-2n$ , where  $n = \dim_{\mathbb{C}} \mathcal{X}$ .

Now assume that the coarse moduli space of  $\mathcal{X}$  is projective. The *genus zero orbifold Gromov-Witten invariants* are integrals of the form:

$$(21) \quad \left\langle \alpha_1 \psi^{k_1}, \dots, \alpha_l \psi^{k_l} \right\rangle_{0,l,d}^{\mathcal{X}} = \int_{[\mathcal{X}_{0,l,d}]^{\text{vir}}} \prod_{i=1}^l \text{ev}_i^*(\alpha_i) \psi_i^{k_i}$$

for  $\alpha_i \in H_{\text{orb}}^*(\mathcal{X})$ ,  $d \in H_2(\mathcal{X}, \mathbb{Q})$  and non-negative integers  $k_i$ . Here  $\mathcal{X}_{0,l,d}$  is the moduli space of (balanced twisted) stable maps to  $\mathcal{X}$  of degree  $d$  and with  $l$  marked points and  $[\mathcal{X}_{0,l,d}]$  is its virtual fundamental class. The map  $\text{ev}_i: \mathcal{X}_{0,l,d} \rightarrow I\mathcal{X}$  is the evaluation map<sup>2</sup> at the  $i$ -th marked point and  $\psi_i$  is the first Chern class of the line bundle over  $\mathcal{X}_{0,l,d}$  whose fiber at a stable map is the cotangent space of the coarse curve at the  $i$ -th marked point. See [1] for details. The Gromov-Witten invariants (21) are non-zero only when  $d$  is in the semigroup  $\text{Eff}_{\mathcal{X}} \subset H_2(\mathcal{X}, \mathbb{Q})$  generated by effective curves.

The *orbifold quantum cohomology* is a formal family of associative and commutative products  $\bullet_{\tau}$  on  $H_{\text{orb}}^*(\mathcal{X}) \otimes \mathbb{C}[[\text{Eff}_{\mathcal{X}}]]$  parametrized by  $\tau \in H_{\text{orb}}^*(\mathcal{X})$ . It is defined by the formula

$$(\alpha \bullet_{\tau} \beta, \gamma)_{\text{orb}} = \sum_{d \in \text{Eff}_{\mathcal{X}}} \sum_{l \geq 0} \sum_{k=1}^N \frac{1}{l!} \langle \alpha, \beta, \tau, \dots, \tau, \gamma \rangle_{0,l+3,d}^{\mathcal{X}} Q^d,$$

where  $Q^d$  is an element of the group ring  $\mathbb{C}[\text{Eff}_{\mathcal{X}}]$  corresponding to  $d \in \text{Eff}_{\mathcal{X}}$ . We decompose the parameter  $\tau$  as

$$(22) \quad \tau = \tau_{0,2} + \tau', \quad \tau_{0,2} \in H^2(\mathcal{X}), \quad \tau' \in \bigoplus_{k \neq 1} H^{2k}(\mathcal{X}) \oplus \bigoplus_{v \in \mathbb{T}'} H^*(\mathcal{X}_v).$$

Using the divisor equation [39, 1], we have

$$(23) \quad (\alpha \bullet_{\tau} \beta, \gamma)_{\text{orb}} = \sum_{d \in \text{Eff}_{\mathcal{X}}} \sum_{l \geq 0} \sum_{k=1}^N \frac{1}{l!} \langle \alpha, \beta, \tau', \dots, \tau', \gamma \rangle_{0,l+3,d}^{\mathcal{X}} e^{\langle \tau_{0,2}, d \rangle} Q^d.$$

Thus the quantum product is a formal power series in  $e^{\tau_{0,2}} Q$  and  $\tau'$ .

**Assumption 3.1.** The specialization  $Q = 1$  of the quantum product  $\bullet_{\tau}$

$$\circ_{\tau} := \bullet_{\tau}|_{Q=1}$$

is convergent over a connected, simply connected open set  $U \subset H_{\text{orb}}^*(\mathcal{X})$  containing the set

$$\{\tau \in H_{\text{orb}}^*(\mathcal{X}) ; \Re \langle \tau_{0,2}, d \rangle < -M, \forall d \in \text{Eff}_{\mathcal{X}} \setminus \{0\}, \|\tau'\| < 1/M\}$$

<sup>2</sup>The map  $\text{ev}_i$  only exists as a map of topological spaces. In [1],  $\text{ev}_i$  takes values in the *rigidified inertia stack* which is the same as  $I\mathcal{X}$  as a topological space but is different as a stack.

for a sufficiently big  $M > 0$ . Here we used the decomposition (22) and  $\|\cdot\|$  is some norm on  $H_{\text{orb}}^*(\mathcal{X})$ .

Under this assumption,  $(H_{\text{orb}}^*(\mathcal{X}), \circ_\tau)$  defines an analytic family of rings over  $U$ . The domain  $U$  here contains the following limit direction:

$$(24) \quad \Re\langle \tau_{0,2}, d \rangle \rightarrow -\infty, \quad \forall d \in \text{Eff}_{\mathcal{X}} \setminus \{0\}, \quad \tau' \rightarrow 0.$$

This is called the *large radius limit*. In this limit,  $\circ_\tau$  goes to the orbifold cup product due to Chen-Ruan [9] (which is the same as the cup product when  $\mathcal{X}$  is a manifold).

**3.2. Quantum cohomology  $\frac{\infty}{2}$ VHS and the Galois action.** Take a homogeneous basis  $\{\phi_i\}_{i=1}^N$  of  $H_{\text{orb}}^*(\mathcal{X})$ . Let  $\{t^i\}_{i=1}^N$  be the linear co-ordinates on  $H_{\text{orb}}^*(\mathcal{X})$  dual to  $\{\phi_i\}_{i=1}^N$ . Let  $\pi: U \times \mathbb{C} \rightarrow U$  be the projection, where  $U \subset H_{\text{orb}}^*(\mathcal{X})$  is the open subset in Assumption 3.1.

**Definition 3.2.** A  $\frac{\infty}{2}$ VHS  $\tilde{\mathcal{F}}$  over  $U$  is defined to be the  $\pi_*\mathcal{O}_{U \times \mathbb{C}}$ -module:

$$\tilde{\mathcal{F}} := H_{\text{orb}}^*(\mathcal{X}) \otimes \pi_*\mathcal{O}_{U \times \mathbb{C}}$$

endowed with the flat connection  $\nabla$  (Dubrovin connection) and a pairing  $(\cdot, \cdot)_{\tilde{\mathcal{F}}}$

$$(25) \quad \nabla := d + \frac{1}{z} \sum_{i=1}^N (\phi_i \circ_\tau) dt^i, \quad (f, g)_{\tilde{\mathcal{F}}} := (f(-z), g(z))_{\text{orb}}.$$

It is graded by the grading operator  $\text{Gr}$  and the Euler vector field  $E$ :

$$\text{Gr} := 2z\partial_z + 2E + 2(\mu + \frac{n}{2}), \quad E := \sum_{i=1}^N (1 - \frac{1}{2} \deg \phi_i) t^i \frac{\partial}{\partial t^i} + \sum_{i=1}^N r^i \frac{\partial}{\partial t^i},$$

where  $n = \dim_{\mathbb{C}} \mathcal{X}$ ,  $c_1(T\mathcal{X}) = \sum_i r^i \phi_i \in H^2(\mathcal{X})$  and  $\mu \in \text{End}(H_{\text{orb}}^*(\mathcal{X}))$  is defined by

$$(26) \quad \mu(\phi_i) := \left( \frac{\deg \phi_i}{2} - \frac{n}{2} \right) \phi_i.$$

The  $\frac{\infty}{2}$ VHS  $\tilde{\mathcal{F}}$  is referred to as *quantum D-module* in the literature [17, 18, 20, 24]. The standard argument (as in [13, 29]) and the WDVV equation in orbifold Gromov-Witten theory [1] show that the Dubrovin connection is flat and that the above data satisfy the axioms of a graded  $\frac{\infty}{2}$ VHS.  $\square$

Let  $H^2(\mathcal{X}, \mathbb{Z})$  denote the cohomology of the constant sheaf  $\mathbb{Z}$  on the topological *stack*  $\mathcal{X}$  (not on the topological *space*). This group is the set of isomorphism classes of topological orbifold line bundles on  $\mathcal{X}$ . Let  $L_\xi \rightarrow \mathcal{X}$  be the orbifold line bundle corresponding to  $\xi \in H^2(\mathcal{X}, \mathbb{Z})$ . Let  $0 \leq f_v(\xi) < 1$  be the rational number such that the stabilizer of  $\mathcal{X}_v$  ( $v \in \mathbb{T}$ ) acts on  $L_\xi|_{\mathcal{X}_v}$  by a complex number  $\exp(2\pi \mathbf{i} f_v(\xi))$ . This number  $f_v(\xi)$  is called the *age* of  $L_\xi$  along  $\mathcal{X}_v$ . Define  $G(\xi): H_{\text{orb}}^*(\mathcal{X}) \rightarrow H_{\text{orb}}^*(\mathcal{X})$  and its derivative  $dG(\xi)$  by

$$\begin{aligned} G(\xi)(\tau_0 \oplus \bigoplus_{v \in \mathbb{T}'} \tau_v) &= (\tau_0 - 2\pi \mathbf{i} \xi_0) \oplus \bigoplus_{v \in \mathbb{T}'} e^{2\pi \mathbf{i} f_v(\xi)} \tau_v, \\ dG(\xi)(\tau_0 \oplus \bigoplus_{v \in \mathbb{T}'} \tau_v) &= \tau_0 \oplus \bigoplus_{v \in \mathbb{T}'} e^{2\pi \mathbf{i} f_v(\xi)} \tau_v, \end{aligned}$$

where  $\tau_v \in H^*(\mathcal{X}_v)$  and  $\xi_0$  is the image of  $\xi$  in  $H^2(\mathcal{X}, \mathbb{Q})$ .

**Proposition 3.3** ([24, Proposition 2.3]). *For  $\xi \in H^2(\mathcal{X}, \mathbb{Z})$ . the map*

$$dG(\xi): \tilde{\mathcal{F}} \rightarrow G(\xi)^* \tilde{\mathcal{F}}, \quad \tilde{\mathcal{F}}_\tau \ni s(z) \mapsto dG(\xi)s(z) \in \tilde{\mathcal{F}}_{G(\xi)\tau}$$

*is a homomorphism of graded  $\frac{\infty}{2}$  VHS's. We call this the Galois action of  $H^2(\mathcal{X}, \mathbb{Z})$  on  $\tilde{\mathcal{F}}$ .*

We can assume that  $U$  is invariant under the Galois action.

**Definition 3.4.** The *quantum cohomology  $\frac{\infty}{2}$  VHS  $\mathcal{F}$*  over  $U/H^2(\mathcal{X}, \mathbb{Z})$  is the quotient of  $\tilde{\mathcal{F}} \rightarrow U$  by the Galois action by  $H^2(\mathcal{X}, \mathbb{Z})$  in Proposition 3.3:

$$\mathcal{F} := (\tilde{\mathcal{F}} \rightarrow U)/H^2(\mathcal{X}, \mathbb{Z}).$$

The flat connection, the pairing and the grading operator on  $\tilde{\mathcal{F}}$  induce those on  $\mathcal{F}$ .  $\square$

**3.3. The fundamental solution  $L(\tau, z)$ .** As in Section 2.1, the graded  $\frac{\infty}{2}$  VHS  $\tilde{\mathcal{F}}$  is rephrased as a flat connection  $\hat{\nabla}$  on the locally free sheaf  $\mathcal{R}^{(0)} = H_{\text{orb}}^*(\mathcal{X}) \otimes \mathcal{O}_{U \times \mathbb{C}^*}$ . Then  $\hat{\nabla}$  defines a  $\mathbb{C}$ -local system  $R = \text{Ker}(\hat{\nabla})$  over  $U \times \mathbb{C}^*$ . A section of the local system  $R$  is a cohomology-valued function  $s(\tau, z)$  satisfying the differential equations:

$$(27) \quad \nabla_k s = \hat{\nabla}_k s = \frac{\partial s}{\partial t^k} + \frac{1}{z} \phi_k \circ_\tau s = 0, \quad k = 1, \dots, N,$$

$$(28) \quad \hat{\nabla}_{z \partial_z} s = z \frac{\partial s}{\partial z} - \frac{1}{z} E \circ_\tau s + \mu s = 0.$$

These equations are called *quantum differential equations*. We give a fundamental solution  $L(\tau, z)$  to the differential equations (27) using gravitational descendants. Let  $\text{pr}: I\mathcal{X} \rightarrow \mathcal{X}$  be the natural projection. We define the action of a class  $\tau_0 \in H^*(\mathcal{X})$  on  $H_{\text{orb}}^*(\mathcal{X})$  by

$$\tau_0 \cdot \alpha = \text{pr}^*(\tau_0) \cup \alpha, \quad \alpha \in H_{\text{orb}}^*(\mathcal{X}),$$

where the right-hand side is the cup product on  $I\mathcal{X}$ . Let  $\{\phi_i\}_{i=1}^N, \{\phi^i\}_{i=1}^N$  be mutually dual bases with respect to the orbifold Poincaré pairing, *i.e.*  $(\phi_i, \phi^j)_{\text{orb}} = \delta_{ij}$ . We define an  $\text{End}(H_{\text{orb}}^*(\mathcal{X}))$ -valued function  $L(\tau, z)$  by

$$(29) \quad L(\tau, z)\phi_i := e^{-\tau_0, 2/z} \phi_i + \sum_{\substack{(d,l) \neq (0,0) \\ d \in \text{Eff}_{\mathcal{X}}}} \sum_{k=1}^N \frac{\phi^k}{k!} \left\langle \phi_k, \tau', \dots, \tau', \frac{e^{-\tau_0, 2/z} \phi_i}{-z - \psi} \right\rangle_{0, l+2, d}^{\mathcal{X}} e^{\langle \tau_0, 2, d \rangle},$$

where we used the decomposition (22) and  $1/(-z - \psi)$  in the correlator should be expanded in the series  $\sum_{k=0}^{\infty} (-z)^{-k-1} \psi^k$ .

**Proposition 3.5** ([24, Proposition 2.4]).  *$L(\tau, z)$  satisfies the following differential equations:*

$$(30) \quad \begin{aligned} \nabla_k L(\tau, z)\phi_i &= 0, \quad k = 1, \dots, N, \\ \hat{\nabla}_{z \partial_z} L(\tau, z)\phi_i &= L(\tau, z)(\mu \phi_i - \frac{\rho}{z} \phi_i), \end{aligned}$$

where  $\rho := c_1(T\mathcal{X}) \in H^2(\mathcal{X})$ . The  $\nabla$ -flat section  $L(\tau, z)\phi_i$  is characterized by the asymptotic initial condition

$$L(\tau, z)\phi_i \sim e^{-\tau_0, 2/z} \phi_i$$



near the large radius limit (24) with  $\tau' = 0$ . Set

$$z^{-\mu} z^\rho := \exp(-\mu \log z) \exp(\rho \log z).$$

Then we have

$$(31) \quad \nabla_k(L(\tau, z) z^{-\mu} z^\rho \phi_i) = 0, \quad \widehat{\nabla}_{z\partial_z}(L(\tau, z) z^{-\mu} z^\rho \phi_i) = 0,$$

$$(32) \quad (L(\tau, -z)\phi_i, L(\tau, z)\phi_j)_{\text{orb}} = (\phi_i, \phi_j)_{\text{orb}},$$

$$(33) \quad dG(\xi)L(G(\xi)^{-1}\tau, z)\alpha = L(\tau, z)e^{-2\pi\mathbf{i}\xi_0/z}e^{2\pi\mathbf{i}f_v(\xi)}\alpha, \quad \alpha \in H^*(\mathcal{X}_v),$$

where  $(dG(\xi), G(\xi))$  is the Galois action associated to  $\xi \in H^2(\mathcal{X}, \mathbb{Z})$ . (See Section 3.2.)

The fundamental solution  $L(\tau, z)$  is a priori formal power series. Under Assumption 3.1, however, the convergence of  $L(\tau, z)$  follows from the fact that it is a solution to the analytic differential equations.

**3.4. The space of (multi-valued) flat sections.** Here we apply the abstract constructions in Section 2.2 to the case of the quantum cohomology  $\frac{\infty}{2}\text{VHS}$ . Using the fundamental solution above, we will identify the spaces  $\mathcal{H}$  and  $\mathcal{V}$  with the Givental's loop space  $\mathcal{H}^\mathcal{X}$  [11] and the cohomology group  $\mathcal{V}^\mathcal{X}$

$$\mathcal{H}^\mathcal{X} := H_{\text{orb}}^*(\mathcal{X}) \otimes \mathcal{O}(\mathbb{C}^*), \quad \mathcal{V}^\mathcal{X} := H_{\text{orb}}^*(\mathcal{X}).$$

For the quantum cohomology  $\frac{\infty}{2}\text{VHS}$ ,  $\mathcal{H}$  (resp.  $\mathcal{V}$ ) consists of cohomology-valued functions  $s(\tau, z)$  satisfying (27) (resp. both (27) and (28)), so we can identify it with  $\mathcal{H}^\mathcal{X}$  (resp.  $\mathcal{V}^\mathcal{X}$ ) via  $L(\tau, z)$  (by (30), (31)):

$$\begin{aligned} \mathcal{H}^\mathcal{X} &\cong \mathcal{H}, \quad \alpha(z) \mapsto L(\tau, z)\alpha(z), \\ \mathcal{V}^\mathcal{X} &\cong \mathcal{V}, \quad \alpha \mapsto L(\tau, z)z^{-\mu}z^\rho\alpha. \end{aligned}$$

These identifications are understood throughout the paper<sup>3</sup>. The flat connection  $\widehat{\nabla}$  and the pairing  $(\cdot, \cdot)_{\widehat{\mathcal{F}}}$  of the quantum cohomology  $\frac{\infty}{2}\text{VHS}$  induces the operator (by (30))

$$(34) \quad \widehat{\nabla}_{z\partial_z}: \mathcal{H}^\mathcal{X} \rightarrow \mathcal{H}^\mathcal{X}, \quad \widehat{\nabla}_{z\partial_z} = z\frac{\partial}{\partial z} + \mu - \frac{\rho}{z}$$

and the pairing (by (32))

$$(35) \quad (\cdot, \cdot)_{\mathcal{H}^\mathcal{X}}: \mathcal{H}^\mathcal{X} \times \mathcal{H}^\mathcal{X} \rightarrow \mathcal{O}(\mathbb{C}^*), \quad (\alpha, \beta)_{\mathcal{H}^\mathcal{X}} = (\alpha(-z), \beta(z))_{\text{orb}}.$$

As in Section 2.2, we can regard  $\mathcal{H}^\mathcal{X}$  as the flat vector bundle  $(\mathcal{H}^\mathcal{X}, \widehat{\nabla}_{z\partial_z})$ :

$$\mathcal{H}^\mathcal{X} := H_{\text{orb}}^*(\mathcal{X}) \times \mathbb{C}^* \rightarrow \mathbb{C}^*, \quad \widehat{\nabla}_{z\partial_z} = z\frac{\partial}{\partial z} + \mu - \frac{\rho}{z}.$$

Then  $\mathcal{V}^\mathcal{X}$  can be identified with the space of multi-valued flat sections of  $\mathcal{H}^\mathcal{X}$ :

$$(36) \quad z^{-\mu}z^\rho: \mathcal{V}^\mathcal{X} \rightarrow \Gamma(\widehat{\mathbb{C}}^*, \mathcal{O}(\mathcal{H}^\mathcal{X})), \quad \alpha \mapsto z^{-\mu}z^\rho\alpha.$$

The pairing  $(\cdot, \cdot)_{\mathcal{V}^\mathcal{X}}$  on  $\mathcal{V}^\mathcal{X}$  (see (7) for the pairing on  $\mathcal{V}$ ) can be written as

$$(37) \quad (\alpha, \beta)_{\mathcal{V}^\mathcal{X}} = (e^{\pi\mathbf{i}\rho}\alpha, e^{\pi\mathbf{i}\mu}\beta)_{\text{orb}}.$$

<sup>3</sup>However, elements of  $\mathcal{H}^\mathcal{X}$  (or  $\mathcal{V}^\mathcal{X}$ ) themselves are loops in the cohomology group (or cohomology classes) and are not treated as flat sections. When we refer to the corresponding sections, we explicitly denote them by  $L(\tau, z)\alpha(z)$  (or  $L(\tau, z)z^{-\mu}z^\rho\alpha$ ) for  $\alpha(z) \in \mathcal{H}^\mathcal{X}$  (or  $\alpha \in \mathcal{V}^\mathcal{X}$ ).

The embedding  $\mathbb{J}_\tau: \widetilde{\mathcal{F}}_\tau \hookrightarrow \mathcal{H}^\mathcal{X}$  of a fiber  $\widetilde{\mathcal{F}}_\tau$  (see (8)) is given by the inverse of  $L(\tau, z)$ :

$$(38) \quad \begin{aligned} \mathbb{J}_\tau \alpha &= L(\tau, z)^{-1} \alpha = L(\tau, -z)^\dagger \alpha \\ &= e^{\tau_{0,2}/z} \left( \alpha + \sum_{\substack{(d,l) \neq (0,0) \\ d \in \text{Eff}_\mathcal{X}}} \sum_{i=1}^N \frac{1}{l!} \left\langle \alpha, \tau', \dots, \tau', \frac{\phi_i}{z - \psi} \right\rangle_{0,l+2,d}^\mathcal{X} e^{\langle \tau_{0,2}, d \rangle} \phi^i \right), \end{aligned}$$

where  $L(\tau, -z)^\dagger$  is the adjoint of  $L(\tau, -z)$  with respect to  $(\cdot, \cdot)_{\text{orb}}$ . The second line follows from (29) and an easy computation of the adjoint  $L(\tau, -z)^\dagger$ . The image  $\mathbb{J}_\tau \mathbf{1}$  of the unit section  $\mathbf{1}$  is called the *J-function*. The image  $\mathbb{F}_\tau = \mathbb{J}_\tau(H_{\text{orb}}^*(\mathcal{X}) \otimes \mathcal{O}(\mathbb{C}))$  of the embedding gives a moving subspace realization of the quantum cohomology  $\frac{\infty}{2}\text{VHS}$ .

The Galois action on  $\widetilde{\mathcal{F}}$  acts on  $\nabla$ -flat sections as  $s(\tau, z) \mapsto dG(\xi)s(G(\xi)^{-1}\tau, z)$ . The following lemma follows from (33).

**Lemma 3.6.** *The Galois actions on  $\mathcal{H}^\mathcal{X}$  and  $\mathcal{V}^\mathcal{X}$  are given by the maps:*

$$(39) \quad G^\mathcal{H}(\xi)(\tau_0 \oplus \bigoplus_{v \in \mathbf{T}'} \tau_v) = e^{-2\pi \mathbf{i} \xi_0/z} \tau_0 \oplus \bigoplus_{v \in \mathbf{T}'} e^{-2\pi \mathbf{i} \xi_0/z} e^{2\pi \mathbf{i} f_v(\xi)} \tau_v,$$

$$(40) \quad G^\mathcal{V}(\xi)(\tau_0 \oplus \bigoplus_{v \in \mathbf{T}'} \tau_v) = e^{-2\pi \mathbf{i} \xi_0} \tau_0 \oplus \bigoplus_{v \in \mathbf{T}'} e^{-2\pi \mathbf{i} \xi_0} e^{2\pi \mathbf{i} f_v(\xi)} \tau_v,$$

where we used the decomposition  $\mathcal{H}^\mathcal{X} = \bigoplus_{v \in \mathbf{T}} H^*(\mathcal{X}_v) \otimes \mathcal{O}(\mathbb{C}^*)$ .

The Galois actions on  $\mathcal{H}^\mathcal{X}$ ,  $\mathcal{V}^\mathcal{X}$  can be viewed as the monodromy of  $\nabla$  over  $U/H^2(\mathcal{X}, \mathbb{Z})$ . The monodromy transformation of  $\widehat{\nabla}_{z\partial_z}$  on  $\mathbb{C}^*$  is given by

$$(41) \quad e^{-2\pi \mathbf{i} \mu} e^{2\pi \mathbf{i} \rho}: \mathcal{V}^\mathcal{X} \longrightarrow \mathcal{V}^\mathcal{X}.$$

This coincides with the Galois action  $(-1)^n G^\mathcal{V}([K_\mathcal{X}])$ . Here,  $[K_\mathcal{X}]$  is the class of the canonical line bundle. When  $\mathcal{X}$  is Calabi-Yau, *i.e.*  $K_\mathcal{X}$  is trivial, the pairing  $(\cdot, \cdot)_{\mathcal{V}^\mathcal{X}}$  is either symmetric or anti-symmetric depending on whether  $n$  is even or odd. In general, this pairing is neither symmetric nor anti-symmetric.

Recall that a real structure on the  $\frac{\infty}{2}\text{VHS}$   $\mathcal{F}$  is given by a sub  $\mathbb{R}$ -local system  $R_\mathbb{R}$  of the  $\mathbb{C}$ -local system  $R$  defined by  $\widehat{\nabla}$  (Definition 2.2). Therefore, a real structure on the quantum cohomology  $\frac{\infty}{2}\text{VHS}$   $\mathcal{F}$  is identified with a monodromy-invariant real subspace  $\mathcal{V}_\mathbb{R}^\mathcal{X}$  in the space  $\mathcal{V}^\mathcal{X}$  of multi-valued  $\widehat{\nabla}$ -flat sections.

**Proposition 3.7.** *A real (integral) structure  $R_\mathbb{R}$  on the quantum cohomology  $\frac{\infty}{2}\text{VHS}$   $\mathcal{F}$  is given by a real subspace  $\mathcal{V}_\mathbb{R}^\mathcal{X}$  (resp. integral lattice  $\mathcal{V}_\mathbb{Z}^\mathcal{X}$ ) of  $\mathcal{V}^\mathcal{X} = H_{\text{orb}}^*(\mathcal{X})$  satisfying*

- (i)  $\mathcal{V}^\mathcal{X} = \mathcal{V}_\mathbb{R}^\mathcal{X} \otimes_\mathbb{R} \mathbb{C}$  (resp.  $\mathcal{V}^\mathcal{X} = \mathcal{V}_\mathbb{Z}^\mathcal{X} \otimes_\mathbb{Z} \mathbb{C}$ );
- (ii)  $\mathcal{V}_\mathbb{R}^\mathcal{X}$  (resp.  $\mathcal{V}_\mathbb{Z}^\mathcal{X}$ ) is invariant under the Galois action (40);
- (iii) The pairing (37) restricted on  $\mathcal{V}_\mathbb{R}^\mathcal{X}$  (resp.  $\mathcal{V}_\mathbb{Z}^\mathcal{X}$ ) takes values in  $\mathbb{R}$  (resp. takes values in  $\mathbb{Z}$  and is unimodular).

A real structure  $R_\mathbb{R}$  on the quantum cohomology  $\frac{\infty}{2}\text{VHS}$  induces the real subspace  $\mathcal{H}_\mathbb{R}^\mathcal{X}$  of  $\mathcal{H}^\mathcal{X}$  (see (9)):

$$\mathcal{H}_\mathbb{R}^\mathcal{X} := \left\{ \alpha(z) \in \mathcal{H}^\mathcal{X} ; \begin{array}{l} L(\tau, \alpha)\alpha(z) \text{ belongs to the fiber of the} \\ \mathbb{R}\text{-local system } R_\mathbb{R} \text{ at each } (\tau, z) \in U \times S^1. \end{array} \right\}.$$

Note that for  $\alpha(z) \in \mathcal{H}^\mathcal{X}$ ,  $L(\tau, z)\alpha(z)$  is not necessarily a section of  $R_\mathbb{R}|_{U \times S^1}$  since it may not be flat. ( $\widehat{\nabla}_{z\partial_z}$  may have monodromy.) Let  $\kappa_\mathcal{H}$  and  $\kappa_\mathcal{V}$  denote the real involutions of  $\mathcal{H}^\mathcal{X}$  and  $\mathcal{V}^\mathcal{X}$  introduced in Section 2.2. We decompose the Galois action on  $\mathcal{H}^\mathcal{X}$  as

$$G^\mathcal{H}(\xi) = e^{-2\pi\mathbf{i}\xi_0/z} G_0^\mathcal{H}(\xi), \quad G_0^\mathcal{H}(\xi) := \bigoplus_{v \in \mathbb{T}} e^{2\pi\mathbf{i}f_v(\xi)}.$$

**Proposition 3.8.** *For any real structure on the quantum cohomology  $\frac{\infty}{2}$  VHS  $\mathcal{F}$ , the following holds. For a real class  $\tau_{0,2} \in H^2(\mathcal{X}, \mathbb{R})$ , we have*

$$(42) \quad \kappa_\mathcal{H}(\tau_{0,2}/z) + (\tau_{0,2}/z)\kappa_\mathcal{H} = 0, \quad \kappa_\mathcal{V}\tau_{0,2} + \tau_{0,2}\kappa_\mathcal{V} = 0,$$

$$(43) \quad G_0^\mathcal{H}(\xi)\kappa_\mathcal{H} = \kappa_\mathcal{H}G_0^\mathcal{H}(\xi),$$

$$(44) \quad (z\partial_z + \mu)\kappa_\mathcal{H} + \kappa_\mathcal{H}(z\partial_z + \mu) = 0,$$

$$(45) \quad \kappa_\mathcal{H} = z^{-\mu}\kappa_\mathcal{V}z^\mu,$$

where the last equality holds when we regard an element of  $\mathcal{H}^\mathcal{X}$  as a  $\mathcal{V}^\mathcal{X}$ -valued function over  $S^1 = \{|z| = 1\}$ . Moreover, if  $\mathcal{X}$  satisfies the following condition:

$$(46) \quad f_v(\xi) = f_{v'}(\xi), \quad \forall \xi \in H^2(\mathcal{X}, \mathbb{Z}) \implies v = v',$$

then we have

$$(47) \quad \begin{aligned} \kappa_\mathcal{H}(H^*(\mathcal{X}_v) \otimes \mathcal{O}(\mathbb{C}^*)) &= H^*(\mathcal{X}_{\text{inv}(v)}) \otimes \mathcal{O}(\mathbb{C}^*), \\ \kappa_\mathcal{V}(H^*(\mathcal{X}_v)) &= H^*(\mathcal{X}_{\text{inv}(v)}). \end{aligned}$$

When (47) holds,  $\kappa_\mathcal{V}$  satisfies

$$(48) \quad \kappa_\mathcal{V}(\alpha) \in \mathcal{C}(\alpha) + H^{>2k}(\mathcal{X}_{\text{inv}(v)}), \quad \alpha \in H^{2k}(\mathcal{X}_v)$$

for some complex antilinear isomorphism  $\mathcal{C} : H^{2k}(\mathcal{X}_v) \rightarrow H^{2k}(\mathcal{X}_{\text{inv}(v)})$ .

*Proof.* Because all the  $f_v(\xi)$ 's are rational numbers, we can find an integer  $m > 0$  such that  $(G_0^\mathcal{H}(\xi))^m = \text{id}$ . Then  $(G^\mathcal{H}(\xi))^m = e^{-2\pi\mathbf{i}m\xi_0/z}$ . Because the Galois action preserves the real structure,  $\xi_0/z$  has to be purely imaginary on  $\mathcal{H}^\mathcal{X}$ . Hence  $\tau_{0,2}/z$  is purely imaginary on  $\mathcal{H}^\mathcal{X}$  for every  $\tau_{0,2} \in H^2(\mathcal{X}, \mathbb{R})$ . From this and (36), we find that the multiplication by  $\tau_{0,2}$  is purely imaginary on  $\mathcal{V}^\mathcal{X}$ . Thus we have (42). From  $G_0^\mathcal{H}(\xi) = e^{2\pi\mathbf{i}\xi_0/z} G^\mathcal{H}(\xi)$ , we have (43). Because  $\widehat{\nabla}_{z\partial_z} = z\partial_z + \mu - \rho/z$  is purely imaginary on  $\mathcal{H}^\mathcal{X}$  (10) and so is  $\rho/z$ , we have (44). By (36),  $\kappa_\mathcal{H}$  and  $\kappa_\mathcal{V}$  are related by

$$\kappa_\mathcal{H} = z^{-\mu}z^\rho\kappa_\mathcal{V}z^{-\rho}z^\mu,$$

where  $z$  is assumed to be in  $S^1$  and both hand sides act on  $\mathcal{V}^\mathcal{X}$ -valued functions over  $S^1$ . Since  $z^\rho = \exp(\rho \log z)$  is real on  $\mathcal{V}^\mathcal{X}$  when  $z \in S^1$ , we have (45). Under the condition (46), the decomposition  $\mathcal{H}^\mathcal{X} = \bigoplus_{v \in \mathbb{T}} H^*(\mathcal{X}_v) \otimes \mathcal{O}(\mathbb{C}^*)$  is the simultaneous eigenspace decomposition for  $G_0^\mathcal{H}(\xi)$ ,  $\xi \in H_2(\mathcal{X}, \mathbb{Z})$ . Therefore, (47) follows from  $\overline{e^{2\pi\mathbf{i}f_v(\xi)}} = e^{2\pi\mathbf{i}f_{\text{inv}(v)}(\xi)}$  and the reality of  $G_0^\mathcal{H}(\xi)$ . Let  $\omega$  be a Kähler class on  $\mathcal{X}$ . The action of  $\omega$  on  $H^*(\mathcal{X}_v)$  is nilpotent. In general, a nilpotent operator  $\omega$  on a vector space defines an increasing filtration  $\{W_k\}_{k \in \mathbb{Z}}$  on it, called a *weight filtration*, which is uniquely determined by the conditions:

$$\omega W_k \subset W_{k-2}, \quad \omega^k : \text{Gr}_k^W \cong \text{Gr}_{-k}^W$$

where  $\mathrm{Gr}_k^W = W_k/W_{k-1}$ . By the Lefschetz decomposition, we know that  $W_k = H^{\geq n_v - k}(\mathcal{X}_v)$  in this case ( $n_v := \dim_{\mathbb{C}} \mathcal{X}_v$ ). Since  $\kappa_{\mathcal{V}}$  anti-commutes with  $\omega$  by (42),  $\kappa_{\mathcal{V}}$  preserves this filtration. This shows (48). Here,  $\mathcal{C}$  is the isomorphism on the associated graded quotient induced from  $\kappa_{\mathcal{V}}$ .  $\square$

**3.5. Purity and polarization.** For an arbitrary real structure, we study a behavior of the quantum cohomology  $\frac{\infty}{2}\mathrm{VHS}$   $\mathcal{F}$  near the large radius limit point (24). We show that it is pure and polarized (in the sense of Definitions 2.8, 2.10) under suitable conditions. Recall that when  $\tilde{\mathcal{F}} \rightarrow U$  is pure, this defines a Cecotti-Vafa structure on the vector bundle  $K \rightarrow U$  by Proposition 2.13.

**Theorem 3.9.** *Let  $\mathcal{X}$  be a smooth Deligne-Mumford stack with a projective coarse moduli space. Let  $\mathcal{F}$  be the quantum cohomology  $\frac{\infty}{2}\mathrm{VHS}$  of  $\mathcal{X}$  and take a real structure on  $\mathcal{F}$ . Let  $\omega$  be a Kähler class on  $\mathcal{X}$ .*

(i) *Assume that the real structure satisfies (47). Then  $\mathcal{F}$  is pure at  $\tau = -x\omega$  when the real part  $\Re(x)$  is sufficiently big.*

(ii) *Assume moreover that the real structure satisfies (cf. (48))*

$$(49) \quad \begin{aligned} & \kappa_{\mathcal{V}}(\alpha) \in (-1)^k \mathbb{R}_{>0} \mathrm{inv}^*(\bar{\alpha}) + H^{>2k}(\mathcal{X}_{\mathrm{inv}(v)}), \\ & \text{or equivalently } \kappa_{\mathcal{H}}(\alpha) = (-1)^k \mathbb{R}_{>0} \mathrm{inv}^*(\bar{\alpha}) z^{-2k+n_v} + O(z^{-2k+n_v-1}) \end{aligned}$$

for  $\alpha \in H^{2k}(\mathcal{X}_v) \subset H_{\mathrm{orb}}^*(\mathcal{X})$ ,  $n_v = \dim_{\mathbb{C}} \mathcal{X}_v$ . Then the Hermitian metric  $h(\cdot, \cdot) = g(\kappa(\cdot), \cdot)$  on the vector bundle  $K \rightarrow U$  satisfies

$$(-1)^{\frac{p-q}{2}} h(u, u) > 0, \quad u \in H^{p,q}(\mathcal{X}_v) \subset K_{-x\omega}, \quad u \neq 0$$

for sufficiently big  $\Re(x) > 0$ , where we identify  $K_{\tau}$  with  $\tilde{\mathcal{F}}_{\tau}/z\tilde{\mathcal{F}}_{\tau} \cong H_{\mathrm{orb}}^*(\mathcal{X})$ . In particular, if  $H_{\mathrm{orb}}^*(\mathcal{X})$  consists only of the  $(p, p)$  part, i.e.  $H^{2p}(\mathcal{X}_v) = H^{p,p}(\mathcal{X}_v)$  for all  $v \in \mathbb{T}$  and  $p \geq 0$ , then  $\mathcal{F}$  is polarized at  $\tau = -x\omega$  for sufficiently big  $\Re(x) > 0$ .

**Remark 3.10.** (i) The condition (47) is satisfied when  $\mathcal{X}$  has enough line bundles to separate the inertia components (see (46) in Proposition 3.8). In particular, (47) is always satisfied when  $\mathcal{X}$  is a manifold.

(ii) We can consider the algebraic quantum cohomology  $\frac{\infty}{2}\mathrm{VHS}$ . Let  $A^*(\mathcal{X})_{\mathbb{C}}$  denote the Chow ring of  $\mathcal{X}$  over  $\mathbb{C}$ . We set  $\mathbb{H}^*(\mathcal{X}_v) := \mathrm{Im}(A^*(\mathcal{X}_v)_{\mathbb{C}} \rightarrow H^*(\mathcal{X}_v))$  and define  $\mathbb{H}_{\mathrm{orb}}^*(\mathcal{X}) := \bigoplus_{v \in \mathbb{T}} \mathbb{H}^*(\mathcal{X}_v)$ . The algebraic quantum cohomology  $\frac{\infty}{2}\mathrm{VHS}$  is defined to be

$$\mathbb{H}_{\mathrm{orb}}^*(\mathcal{X}) \otimes \pi_* \mathcal{O}_{(U \cap \mathbb{H}_{\mathrm{orb}}(\mathcal{X})) \times \mathbb{C}}$$

with the restriction of Dubrovin connection, the grading operator and pairing, modulo the Galois action given by an element of  $\mathrm{Pic}(\mathcal{X})$ . Here we used the fact that the quantum product among classes in  $\mathbb{H}_{\mathrm{orb}}^*(\mathcal{X})$  again belongs to  $\mathbb{H}_{\mathrm{orb}}^*(\mathcal{X})$ . This follows from the algebraic construction of orbifold Gromov-Witten theory [1]. When we assume the Hodge conjecture for all  $\mathcal{X}_v$ , each  $\mathbb{H}^*(\mathcal{X}_v)$  has the Poincaré duality and the orbifold Poincaré pairing is non-degenerate on  $\mathbb{H}_{\mathrm{orb}}^*(\mathcal{X})$ . Under this assumption, the algebraic quantum cohomology  $\frac{\infty}{2}\mathrm{VHS}$  is pure and polarized at  $\tau = -x\omega$  for a Kähler class  $\omega \in \mathbb{H}^2(\mathcal{X})$  and  $\Re(x) \gg 0$  if the conditions corresponding to (47) and (49) are satisfied. The proof below applies to the algebraic quantum cohomology  $\frac{\infty}{2}\mathrm{VHS}$  without change.

Note that the Poincaré duality of  $\mathbb{H}^*(\mathcal{X}_v)$  also implies the Hard Lefschetz of it used in the proof below.

**Remark 3.11.** Hertling [21] and Hertling-Sevenheck [22] studied similar problems for general TERP structures. They considered the change of TERP structures induced by the rescaling  $z \mapsto rz$  of the parameter  $z$ . This rescaling with  $r \rightarrow \infty$  is called Sabbah orbit in [22] and is equivalent to the flow of minus the Euler vector field:  $\tau \mapsto \tau - \rho \log r$  for  $\tau \in H^2(\mathcal{X})$ . When  $\mathcal{X}$  is Fano and  $\omega = c_1(\mathcal{X}) = \rho$ , the large radius limit corresponds to the Sabbah orbit<sup>4</sup>, and the conclusions in Theorem 3.9 can be deduced from [22, Theorem 7.3] in this case.

**Remark 3.12.** Singularity theory gives a  $\frac{\infty}{2}$ VHS with a real structure. According to the recent work of Sabbah [34, Section 4], the  $\frac{\infty}{2}$ VHS arising from a cohomologically tame function on an affine manifold is pure and polarized. In Section 4, we use this result to see that the  $tt^*$ -geometry of  $\mathbb{P}^1$  is pure and polarized everywhere.

The rest of this section is devoted to the proof of Theorem 3.9.

From Equation (38), we see that  $e^{x\omega/z} \mathbb{J}_{-x\omega}(\varphi) \rightarrow \varphi$  as  $\Re(x) \rightarrow \infty$ . Thus, in the moving subspace realization, the Hodge structure  $\mathbb{F}_{-x\omega} = \mathbb{J}_{-x\omega}(\mathcal{F}_{-x\omega})$  has the asymptotics:

$$\mathbb{F}_{-x\omega} \sim e^{-x\omega/z} \mathbb{F}_{\text{lim}} \quad \text{as } \Re(x) \rightarrow \infty,$$

where  $\mathbb{F}_{\text{lim}} := H_{\text{orb}}^*(\mathcal{X}) \otimes \mathcal{O}(\mathbb{C})$  is the limiting Hodge structure. This is an analogue of the *nilpotent orbit theorem* [35] in quantum cohomology. First we study the behavior of the nilpotent orbit  $x \mapsto e^{-x\omega/z} \mathbb{F}_{\text{lim}}$  for  $\Re(x) \gg 0$  (see Proposition 3.15 below).

**(Step 1)** We study the purity of the  $\frac{\infty}{2}$ VHS  $x \mapsto e^{-x\omega/z} \mathbb{F}_{\text{lim}}$ , *i.e.* if the natural map

$$(50) \quad e^{-x\omega/z} \mathbb{F}_{\text{lim}} \cap \kappa_{\mathcal{H}}(e^{-x\omega/z} \mathbb{F}_{\text{lim}}) \longrightarrow e^{-x\omega/z} (\mathbb{F}_{\text{lim}}/z\mathbb{F}_{\text{lim}}) \cong e^{-x\omega/z} H_{\text{orb}}^*(\mathcal{X})$$

is an isomorphism (see (12) in Proposition 2.9). Under the condition (47), this is equivalent to that the map

$$e^{-x\omega/z} H^*(\mathcal{X}_v)\{z\} \cap \kappa_{\mathcal{H}}(e^{-x\omega/z} H^*(\mathcal{X}_{\text{inv}(v)})\{z\}) \rightarrow e^{-x\omega/z} H^*(\mathcal{X}_v)$$

is an isomorphism for each  $v \in \mathbb{T}$ . Here we put  $H^*(\mathcal{X}_v)\{z\} := H^*(\mathcal{X}_v) \otimes \mathcal{O}(\mathbb{C})$ . Since  $\kappa_{\mathcal{H}} e^{-x\omega/z} = e^{\overline{x}\omega/z} \kappa_{\mathcal{H}}$  (see (42)), this is equivalent to that

$$H^*(\mathcal{X}_v)\{z\} \cap e^{2t\omega/z} \kappa_{\mathcal{H}}(H^*(\mathcal{X}_{\text{inv}(v)})\{z\}) \rightarrow H^*(\mathcal{X}_v), \quad t := \Re(x)$$

is an isomorphism. We further decompose this into  $(z\partial_z + \mu)$ -eigenspaces. Because  $z\partial_z + \mu$  is purely imaginary (44), the above map between the  $(z\partial_z + \mu)$ -eigenspaces of the eigenvalue  $\frac{1}{2}(-k + \text{age}(v) - \text{age}(\text{inv}(v)))$  is of the form:

$$\left( \bigoplus_{l \geq 0} H^{n_v - k - 2l}(\mathcal{X}_v) z^l \right) \cap e^{2t\omega/z} \kappa_{\mathcal{H}} \left( \bigoplus_{l \geq 0} H^{n_v + k - 2l}(\mathcal{X}_{\text{inv}(v)}) z^l \right) \rightarrow H^{n_v - k}(\mathcal{X}_v).$$

Here,  $n_v = \dim_{\mathbb{C}} \mathcal{X}_v$  and  $k$  is an integer such that  $n_v - k$  is even. By using (45), we find that this map is conjugate (via  $z^{\mu + (k - \iota_v + \iota_{\text{inv}(v)})/2}$ ) to the following map:

$$(51) \quad H^{\leq n_v - k}(\mathcal{X}_v) \cap e^{2t\omega} \kappa_{\mathcal{V}}(H^{\leq n_v + k}(\mathcal{X}_{\text{inv}(v)})) \rightarrow H^{n_v - k}(\mathcal{X}_v)$$

<sup>4</sup> The author thanks Claus Hertling for this remark.

which is induced by  $H^{\leq n_v - k}(\mathcal{X}_v) \rightarrow H^{\leq n_v - k}(\mathcal{X}_v)/H^{\leq n_v - k - 2}(\mathcal{X}_v) \cong H^{n_v - k}(\mathcal{X}_v)$ . We will show that this becomes an isomorphism for  $t = \Re(x) \gg 0$  in Lemma 3.14 below.

Let  $\mathfrak{a}: H^*(\mathcal{X}_v) \rightarrow H^{*+2}(\mathcal{X}_v)$  be the operator defined by  $\mathfrak{a}(\phi) := \omega \cup \phi$ . There exists an operator  $\mathfrak{a}^\dagger: H^*(\mathcal{X}_v) \rightarrow H^{*-2}(\mathcal{X}_v)$  such that  $\mathfrak{a}$  and  $\mathfrak{a}^\dagger$  generate the Lefschetz  $\mathfrak{sl}_2$ -action on  $H^*(\mathcal{X}_v)$ :

$$[\mathfrak{a}, \mathfrak{a}^\dagger] = h, \quad [h, \mathfrak{a}] = 2\mathfrak{a}, \quad [h, \mathfrak{a}^\dagger] = -2\mathfrak{a}^\dagger,$$

where  $h := \deg - n_v$  is the (shifted) grading operator. Note that  $\mathfrak{a}^\dagger$  is uniquely determined by the above commutation relation and that  $\mathfrak{a}^\dagger$  annihilates the primitive cohomology  $PH^{n_v - k}(\mathcal{X}_v) := \text{Ker}(\mathfrak{a}^{k+1}: H^{n_v - k}(\mathcal{X}_v) \rightarrow H^{n_v + k + 2}(\mathcal{X}_v))$ .

**Lemma 3.13.** *The map  $e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}: H^*(\mathcal{X}_v) \rightarrow H^*(\mathcal{X}_v)$  sends  $H^{\geq n_v - k}(\mathcal{X}_v)$  onto  $H^{\leq n_v + k}(\mathcal{X}_v)$  isomorphically. Moreover, for  $u \in \mathfrak{a}^j PH^{n_v - k - 2j}(\mathcal{X}_v) \subset H^{n_v - k}(\mathcal{X}_v)$ , one has*

$$e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}u = (-1)^{k+j} \frac{j!}{(k+j)!} \omega^k u + H^{< n_v + k}(\mathcal{X}_v).$$

*Proof.* An easy calculation shows that

$$e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}\mathfrak{a} = -\mathfrak{a}^\dagger e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}.$$

Therefore,  $e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}$  should send the weight filtration for the nilpotent operator  $\mathfrak{a}$  to that for  $\mathfrak{a}^\dagger$ . But the weight filtration for  $\mathfrak{a}$  is  $\{H^{\geq n_v - k}\}_k$  and that for  $\mathfrak{a}^\dagger$  is  $\{H^{\leq n_v + k}\}_k$  (see the proof of Proposition 3.8 for weight filtration). Take  $u \in \mathfrak{a}^j PH^{n_v - k - 2j}(\mathcal{X}_v)$ . Put  $u = \mathfrak{a}^j \phi$  for  $\phi \in PH^{n_v - k - 2j}(\mathcal{X}_v)$ . We calculate

$$\begin{aligned} e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}u &= e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}\mathfrak{a}^j\phi = (-\mathfrak{a}^\dagger)^je^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}\phi = (-\mathfrak{a}^\dagger)^je^{-\mathfrak{a}}\phi \\ &= (-\mathfrak{a}^\dagger)^j \frac{(-1)^{k+2j}}{(k+2j)!} \mathfrak{a}^{k+2j}\phi + \text{lower degree term}, \end{aligned}$$

where in the second line we used that  $e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}u \in H^{\leq n_v + k}(\mathcal{X}_v)$ . Using  $\mathfrak{a}^\dagger \mathfrak{a}^l u = l(k + 2j + 1 - l)\mathfrak{a}^{l-1}u$ , we arrive at the formula for  $e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}u$ .  $\square$

**Lemma 3.14.** *The map (51) is an isomorphism for sufficiently big  $t > 0$ . Moreover,  $u \in H^{n_v - k}(\mathcal{X}_v)$  corresponds to an element of the form*

$$(2t)^{(\deg + k - n_v)/2} (e^{\mathfrak{a}^\dagger}u + O(t^{-1})) \in H^{\leq n_v - k}(\mathcal{X}_v) \cap e^{2\omega t} \kappa_{\mathcal{V}}(H^{\leq n_v + k}(\mathcal{X}_{\text{inv}(v)}))$$

under (51), where  $(2t)^{\deg/2}$  is defined by  $(2t)^{\deg/2} = (2t)^k$  on  $H^{2k}(\mathcal{X}_v)$ .

*Proof.* First we rescale (51) by  $(2t)^{-\deg/2}$ :

$$\begin{array}{ccc} H^{\leq n_v - k}(\mathcal{X}_v) \cap e^{2\omega t} \kappa_{\mathcal{V}}(H^{\leq n_v + k}(\mathcal{X}_{\text{inv}(v)})) & \longrightarrow & H^{n_v - k}(\mathcal{X}_v) \\ \downarrow (2t)^{-\deg/2} & & \downarrow (2t)^{-\deg/2} \\ H^{\leq n_v - k}(\mathcal{X}_v) \cap e^{\omega} \kappa_t(H^{\leq n_v + k}(\mathcal{X}_{\text{inv}(v)})) & \longrightarrow & H^{n_v - k}(\mathcal{X}_v), \end{array}$$

where  $\kappa_t := (2t)^{-\deg/2} \kappa_{\mathcal{V}}(2t)^{\deg/2}$ . Since the column arrows are isomorphisms for all  $t \in \mathbb{R}$ , it suffices to show that the bottom arrow is an isomorphism for  $t \gg 0$ . Observe that the expected dimension of  $H^{\leq n_v - k}(\mathcal{X}_v) \cap e^{2\omega} \kappa_t(H^{\leq n_v + k}(\mathcal{X}_{\text{inv}(v)}))$  equals

$\dim H^{n_v-k}(\mathcal{X}_v)$  by Poincaré duality. Thus that the bottom arrow becomes an isomorphism is an open condition for  $\kappa_t$ . By (48) in Proposition 3.8, we have

$$(52) \quad \kappa_t = \mathcal{C} + O(t^{-1}),$$

for a degree preserving isomorphism  $\mathcal{C}: H^*(\mathcal{X}_{\text{inv}(v)}) \cong H^*(\mathcal{X}_v)$ . Therefore, we only need to check that the map at  $t = \infty$

$$(53) \quad H^{\leq n_v-k}(\mathcal{X}_v) \cap e^{\mathfrak{a}} H^{\leq n_v+k}(\mathcal{X}_v) \rightarrow H^{n_v-k}(\mathcal{X}_v)$$

is an isomorphism (recall that  $\mathfrak{a} = \omega \cup$ ). Note that this factors through  $\exp(-\mathfrak{a}^\dagger)$  as

$$H^{\leq n_v-k} \cap e^{\mathfrak{a}} H^{\leq n_v+k} \xrightarrow{\exp(-\mathfrak{a}^\dagger)} H^{\leq n_v-k} \cap e^{-\mathfrak{a}^\dagger} e^{\mathfrak{a}} H^{\leq n_v+k} \longrightarrow H^{n_v-k},$$

where we omitted the space  $\mathcal{X}_v$  from the notation. The second map is induced from the projection  $H^{\leq n_v-k} \rightarrow H^{n_v-k}$  again. Because  $e^{-\mathfrak{a}^\dagger} e^{\mathfrak{a}}(H^{\leq n_v+k}) = H^{\geq n_v-k}$  by Lemma 3.13, we know that the map (53) is an isomorphism and that the inverse map is given by  $u \mapsto \exp(\mathfrak{a}^\dagger)u$ . Now the conclusion follows.  $\square$

**Proposition 3.15.** *Assume that (47) holds. Then the nilpotent orbit  $x \mapsto e^{-x\omega/z} \mathbb{F}_{\text{lim}}$  is pure for sufficiently big  $t = \Re(x) > 0$  i.e. the map (50) is an isomorphism for  $t \gg 0$ . The inverse image of  $e^{-x\omega/z}u$ ,  $u \in H^{n_v-k}(\mathcal{X}_v)$  under (50) is of the form  $e^{-x\omega/z} \varpi_t(u)$  with*

$$(54) \quad \varpi_t(u) = z^{-\mu-(k-\iota_v+\iota_{\text{inv}(v)})/2} (2t)^{(\deg+k-n_v)/2} (e^{\mathfrak{a}^\dagger}u + O(t^{-1})) \in \bigoplus_{l \geq 0} H^{n_v-k-2l}(\mathcal{X}_v) z^l.$$

When  $u = \mathfrak{a}^j \phi$  and  $\phi \in PH^{n_v-k-2j}(\mathcal{X}_v)$ , we have

$$(55) \quad (\kappa_{\mathcal{H}}(e^{-x\omega/z} \varpi_t(u)), e^{-x\omega/z} \varpi_t(u))_{\mathcal{H}^X} = \frac{(2t)^k j!}{(k+j)!} \int_{\mathcal{X}_v} \omega^{k+2j} \phi \cup \text{inv}^* \mathcal{C}(\phi) + O(t^{k-1})$$

where  $\mathcal{C}: H^*(\mathcal{X}_v) \rightarrow H^*(\mathcal{X}_{\text{inv}(v)})$  is the isomorphism appearing in (48) and  $(\cdot, \cdot)_{\mathcal{H}^X}$  is given in (35). If moreover  $u \in H^{p,q}(\mathcal{X}_v) \setminus \{0\}$  and the condition (49) holds,

$$(-1)^{(p-q)/2} (\kappa_{\mathcal{H}}(e^{-x\omega/z} \varpi_t(u)), e^{-x\omega/z} \varpi_t(u))_{\mathcal{H}^X} > 0$$

for  $t = \Re(x) \gg 0$ . (Here  $p+q = n_v - k$ .)

*Proof.* The purity of  $e^{-x\omega/z} \mathbb{F}_{\text{lim}}$  and the formula for  $\varpi_t(u)$  follow from Lemma 3.14 and the discussion preceding (51). Putting  $c = (-k + \iota_v - \iota_{\text{inv}(v)})/2$ , we calculate

$$\begin{aligned} (\kappa_{\mathcal{H}}(e^{-x\omega/z} \varpi_t(u)), e^{-x\omega/z} \varpi_t(u))_{\mathcal{H}^X} &= (\kappa_{\mathcal{H}}(\varpi_t(u)), e^{-2t\omega/z} \varpi_t(u))_{\mathcal{H}^X} \\ &= (2t)^{k-n_v} (z^{-\mu-c} \kappa_{\mathcal{V}}(2t)^{\deg/2} (e^{\mathfrak{a}^\dagger}u + O(t^{-1})), z^{-\mu+c} (2t)^{\deg/2} (e^{-\mathfrak{a}} e^{\mathfrak{a}^\dagger}u + O(t^{-1})))_{\mathcal{H}^X} \\ &= (2t)^k ((-1)^{-\mu-c} \kappa_t(e^{\mathfrak{a}^\dagger}u + O(t^{-1})), e^{-\mathfrak{a}} e^{\mathfrak{a}^\dagger}u + O(t^{-1}))_{\text{orb}} \end{aligned}$$

where we used  $e^{-2t\omega/z} z^{-\mu} (2t)^{\deg/2} = z^{-\mu} (2t)^{\deg/2} e^{-\mathfrak{a}}$  and (45) in the second line (we assume  $|z| = 1$ ) and set  $\kappa_t := (2t)^{-\deg/2} \kappa_{\mathcal{V}}(2t)^{\deg/2}$  again in the third line. From (52), the highest order term in  $t$  becomes

$$(2t)^k ((-1)^{-\mu-c} e^{-\mathfrak{a}^\dagger} \mathcal{C}(u), e^{-\mathfrak{a}} e^{\mathfrak{a}^\dagger} u)_{\text{orb}}.$$

Note that  $\mathcal{C}$  anticommutes with  $\mathfrak{a}, \mathfrak{a}^\dagger$  by (42). By a calculation using Lemma 3.13, we find that this equals the highest order term of the right-hand side of (55). The last statement on positivity follows from the classical Hodge-Riemann bilinear inequality:

$$(-1)^{(p-q)/2}(-1)^{(n_v-k)/2-j} \int_{\mathcal{X}_v} \omega^{k+2j} \phi \cup \bar{\phi} > 0$$

for  $\phi \in PH^{n_v-k-2j}(\mathcal{X}_v) \cap H^{p-j, q-j}(\mathcal{X}_v) \setminus \{0\}$ ,  $n_v - k$  even.  $\square$

**(Step 2)** Next we show that  $x \mapsto \mathbb{F}_{-x\omega}$  is pure for  $t = \Re(x) \gg 0$ . We set  $\mathbb{F}'_{-x\omega} = e^{x\omega/z} \mathbb{F}_{-x\omega}$ . Again by (12) in Proposition 2.9 and  $\kappa_{\mathcal{H}} e^{-x\omega/z} = e^{\bar{x}\omega/z} \kappa_{\mathcal{H}}$ , it is sufficient to show that

$$\mathbb{F}'_{-x\omega} \cap e^{2t\omega/z} \kappa_{\mathcal{H}}(\mathbb{F}'_{-x\omega}) \longrightarrow \mathbb{F}'_{-x\omega} / z \mathbb{F}'_{-x\omega}$$

is an isomorphism. Put  $\kappa^t = e^{2t\omega/z} \kappa_{\mathcal{H}}$  ( $\kappa^t$  is different from  $\kappa_t$  appearing in (52)). Fix a basis  $\{\phi_1, \dots, \phi_N\}$  of  $H_{\text{orb}}^*(\mathcal{X})$ . Define an  $N \times N$  matrix  $A_t(z, z^{-1})$  by

$$(56) \quad [\kappa^t(\phi_1), \dots, \kappa^t(\phi_N)] = [\phi_1, \dots, \phi_N] A_t(z, z^{-1}).$$

This matrix  $A_t$  is a Laurent polynomial in  $z$  (by (45)) and a polynomial in  $t$ . We already showed that (50) is an isomorphism for  $t = \Re(x) \gg 0$ . Therefore,  $A_t(z)$  admits the Birkhoff factorization  $A_t(z) = B_t(z)C_t(z)$  for  $t \gg 0$ , where  $B_t: \mathbb{D}_0 \rightarrow GL_N(\mathbb{C})$  with  $B_t(0) = \mathbf{1}$  and  $C_t: \mathbb{D}_\infty \rightarrow GL_N(\mathbb{C})$  (see Remark 2.11). The matrix  $B_t(z)$  here is given by

$$[\varpi_t(\phi_1), \dots, \varpi_t(\phi_N)] = [\phi_1, \dots, \phi_N] B_t(z)$$

for  $\varpi_t(\phi_i)$  appearing in (54). In particular,  $B_t(z)$  and  $C_t(z)$  are polynomials in  $z$  and  $z^{-1}$  respectively and have at most polynomial growth in  $t$ . We define  $Q_x: \mathbb{P}^1 \setminus \{0\} \rightarrow GL_N(\mathbb{C})$  by

$$(57) \quad [j_1, \dots, j_N] = [\phi_1, \dots, \phi_N] Q_x(z), \quad j_i := e^{x\omega/z} \mathbb{J}_{-x\omega}(\phi_i)$$

where  $\mathbb{J}_\tau$  is given in (38). The vectors  $j_1, \dots, j_N$  form a basis of  $\mathbb{F}'_{-t\omega}$  and  $Q_x(\infty) = \mathbf{1}$ . Note that  $Q_x = \mathbf{1} + O(e^{-\epsilon_0 t})$  as  $t = \Re(x) \rightarrow \infty$  for  $\epsilon_0 := \min(\langle \omega, d \rangle; d \in \text{Eff}_{\mathcal{X}} \setminus \{0\})$ . From (56) and (57), we find

$$[\kappa^t(j_1), \dots, \kappa^t(j_N)] = [j_1, \dots, j_N] Q_x^{-1} A_t \bar{Q}_x,$$

where  $\bar{Q}_x$  is the complex conjugate of  $Q_x$  with  $z$  restricted to  $S^1 = \{|z| = 1\}$ . As we did in Remark 2.11, it suffices to show that  $Q_x^{-1} A_t \bar{Q}_x$  admits the Birkhoff factorization. We have

$$Q_x^{-1} A_t \bar{Q}_x = B_t(B_t^{-1} Q_x^{-1} B_t)(C_t \bar{Q}_x C_t^{-1}) C_t$$

and for  $0 < \epsilon < \epsilon_0$ ,

$$B_t^{-1} Q_x^{-1} B_t = \mathbf{1} + O(e^{-\epsilon t}), \quad C_t \bar{Q}_x C_t^{-1} = \mathbf{1} + O(e^{-\epsilon t}), \quad \text{as } t = \Re(x) \rightarrow \infty.$$

Here we used that  $B_t$  and  $C_t$  have at most polynomial growth in  $t$ . By the continuity of Birkhoff factorization,  $(B_t^{-1} Q_x^{-1} B_t)(C_t \bar{Q}_x C_t^{-1}) = \mathbf{1} + O(e^{-\epsilon t})$  admits the Birkhoff factorization of the form:

$$(58) \quad (B_t^{-1} Q_x^{-1} B_t)(C_t \bar{Q}_x C_t^{-1}) = \tilde{B}_x(z) \tilde{C}_x(z), \quad \tilde{B}_x = \mathbf{1} + O(e^{-\epsilon t}), \quad \tilde{C}_x = \mathbf{1} + O(e^{-\epsilon t})$$

for  $t = \Re(x) \gg 0$ , where  $\tilde{B}_x: \mathbb{D}_0 \rightarrow GL_N(\mathbb{C})$ ,  $\tilde{B}_x(0) = \mathbf{1}$  and  $\tilde{C}_x: \mathbb{D}_\infty \rightarrow GL_N(\mathbb{C})$ . The order estimate  $O(e^{-\epsilon t})$  holds in the  $C^0$ -norm on the loop space  $C^\infty(S^1, \text{End}(\mathbb{C}^N))$ . See



Appendix 5.1 for the proof of the order estimate in (58). Therefore  $Q_x^{-1}A_t\overline{Q}_x$  also has the Birkhoff factorization for  $t = \Re(x) \gg 0$  and we know that

$$[\Pi_x(\phi_1), \dots, \Pi_x(\phi_N)] := [\phi_1, \dots, \phi_N]Q_x(z)B_t(z)\tilde{B}_x(z)$$

form a basis of  $\mathbb{F}'_{-x\omega} \cap \kappa^t(\mathbb{F}'_{-x\omega})$ , i.e.  $e^{-x\omega/z}\Pi_x(\phi_1), \dots, e^{-x\omega/z}\Pi_x(\phi_N)$  form a basis of  $\mathbb{F}_{-x\omega} \cap \kappa\mathcal{H}(\mathbb{F}_{-x\omega})$ . Using that  $\Pi_x(\phi_i) = \varpi_x(\phi_i) + O(e^{-\epsilon t})$  and Proposition 3.15, we have

$$(-1)^{(p-q)/2}(\kappa\mathcal{H}(e^{-x\omega/z}\Pi_x(\phi)), e^{-x\omega/z}\Pi_x(\phi))_{\mathcal{H}^X} > 0, \quad \phi \in H^{p,q}(\mathcal{X}_v) \setminus \{0\}$$

for sufficiently big  $\Re(x) > 0$ . This completes the proof of Theorem 3.9.

**3.6. The  $\widehat{\Gamma}$ -integral (real) structure.** Real or integral structures in the quantum cohomology  $\frac{\infty}{2}$ VHS (in the sense of Definition 2.2) are not unique. In this section, we construct an integral structure ( $\widehat{\Gamma}$ -integral structure) which makes sense for a general symplectic orbifold, using  $K$ -theory. Since this satisfies the assumption of Theorem 3.9, it yields a Cecotti-Vafa structure near the large radius limit point. We showed in [24] that the  $\widehat{\Gamma}$ -integral structure for a weak Fano toric orbifold coincides with the integral structure on the singularity mirror (Landau-Ginzburg model) [17, 18].

Let  $K(\mathcal{X})$  denote the Grothendieck group of topological orbifold vector bundles on  $\mathcal{X}$ . See e.g. [2, 28, 31] for vector bundles on orbifolds. For simplicity, we assume that  $\mathcal{X}$  is isomorphic to the quotient  $[Y/G]$  as a topological orbifold where  $Y$  is a manifold and  $G$  is a compact Lie group. In this case  $K(\mathcal{X})$  is a finitely generated abelian group [2]. For example, an orbifold without generic stabilizers can be presented as a quotient orbifold  $Y/G$  (see e.g. [2]). For an orbifold vector bundle  $\tilde{V}$  on the inertia stack  $I\mathcal{X}$ , we have an eigenbundle decomposition of  $\tilde{V}|_{\mathcal{X}_v}$

$$\tilde{V}|_{\mathcal{X}_v} = \bigoplus_{0 \leq f < 1} \tilde{V}_{v,f}$$

with respect to the stabilizer action over  $\mathcal{X}_v$ . Here, the stabilizer acts on the component  $\tilde{V}_{v,f}$  by  $\exp(2\pi\mathbf{i}f) \in \mathbb{C}$ . Let  $\text{pr}: I\mathcal{X} \rightarrow \mathcal{X}$  be the projection. For an orbifold vector bundle  $V$ , the Chern character  $\tilde{\text{ch}}(V) \in H^*(I\mathcal{X})$  is defined by

$$\tilde{\text{ch}}(V) := \bigoplus_{v \in \mathbb{T}} \sum_{0 \leq f < 1} e^{2\pi\mathbf{i}f} \text{ch}((\text{pr}^* V)_{v,f})$$

where  $\text{ch}$  is the ordinary Chern character and  $V$  is an orbifold vector bundle on  $\mathcal{X}$ . For an orbifold vector bundle  $V$  on  $\mathcal{X}$ , let  $\delta_{v,f,i}$ ,  $i = 1, \dots, l_{v,f}$  be the Chern roots of  $(\text{pr}^* V)_{v,f}$ . The Todd class  $\widetilde{\text{Td}}(V) \in H^*(I\mathcal{X})$  is defined by

$$\widetilde{\text{Td}}(V) = \bigoplus_{v \in \mathbb{T}} \prod_{0 \leq f < 1, 1 \leq i \leq l_{v,f}} \frac{1}{1 - e^{-2\pi\mathbf{i}f} e^{-\delta_{v,f,i}}} \prod_{f=0, 1 \leq i \leq l_{v,0}} \frac{\delta_{v,0,i}}{1 - e^{-\delta_{v,0,i}}}$$

We put  $\widetilde{\text{Td}}_{\mathcal{X}} := \widetilde{\text{Td}}(T\mathcal{X})$ . For a holomorphic orbifold vector bundle  $V$ , the holomorphic Euler characteristic  $\chi(V) := \sum_{i=0}^{\dim \mathcal{X}} (-1)^i \dim H^i(\mathcal{X}, V)$  is given by the Kawasaki-Riemann-Roch formula [27, 38]:

$$(59) \quad \chi(V) = \int_{I\mathcal{X}} \tilde{\text{ch}}(V) \cup \widetilde{\text{Td}}_{\mathcal{X}}.$$

For a not necessarily holomorphic orbifold vector bundle, we can use the right-hand side of (59) as the definition of  $\chi(V)$ . It follows from Kawasaki's  $V$ -index theorem [28] that  $\chi(V)$  is always an integer (see [24, Remark 2.8]). We define a multiplicative characteristic class  $\widehat{\Gamma}: K(\mathcal{X}) \rightarrow H^*(I\mathcal{X})$  called the  $\widehat{\Gamma}$ -class [24, 26] by

$$\widehat{\Gamma}(V) := \bigoplus_{v \in \mathbf{T}} \prod_{0 \leq f < 1} \prod_{i=1}^{l_{v,f}} \Gamma(1 - f + \delta_{v,f,i}),$$

where  $\delta_{v,f,i}$  is the same as above. The Gamma function on the right-hand side should be expanded in series at  $1 - f > 0$ . This class can be regarded as a “half” of the Todd class. When  $\mathcal{X}$  is a manifold  $X$ , by using  $\Gamma(1 - z)\Gamma(1 + z) = e^{-\pi \mathbf{i} z} \pi z / (1 - e^{-2\pi \mathbf{i} z})$ , we have

$$e^{\pi \mathbf{i} c_1(X)} \cup \widehat{\Gamma}(V) \cup (-1)^{\deg/2} \widehat{\Gamma}(V) = (2\pi \mathbf{i})^{\deg/2} \mathrm{Td}(V).$$

**Definition-Proposition 3.16** ([24, Proposition 2.10]). *Put  $\widehat{\Gamma}_{\mathcal{X}} := \widehat{\Gamma}(T\mathcal{X})$ . Define an integral structure  $\mathcal{V}_{\mathbb{Z}}^{\mathcal{X}} \subset \mathcal{V}^{\mathcal{X}} = H_{\mathrm{orb}}^*(\mathcal{X})$  to be the image of the map*

$$(60) \quad \Psi: K(\mathcal{X}) \longrightarrow \mathcal{V}^{\mathcal{X}}, \quad [V] \longmapsto \frac{1}{(2\pi)^{n/2}} \widehat{\Gamma}_{\mathcal{X}} \cup (2\pi \mathbf{i})^{\deg/2} \mathrm{inv}^*(\widetilde{\mathrm{ch}}(V)),$$

where  $\deg: H^*(I\mathcal{X}) \rightarrow H^*(I\mathcal{X})$  is a grading operator on  $H^*(I\mathcal{X})$  defined by  $\deg = 2k$  on  $H^{2k}(I\mathcal{X})$  and  $\cup$  is the cup product in  $H^*(I\mathcal{X})$ . Then

- (i)  $\mathcal{V}_{\mathbb{Z}}^{\mathcal{X}}$  is a lattice in  $\mathcal{V}^{\mathcal{X}}$  such that  $\mathcal{V}^{\mathcal{X}} \cong \mathcal{V}_{\mathbb{Z}}^{\mathcal{X}} \otimes_{\mathbb{Z}} \mathbb{C}$ .
- (ii) The Galois action  $G^{\mathcal{V}}(\xi)$  on  $\mathcal{V}^{\mathcal{X}}$  in (40) corresponds to tensoring by the line bundle  $\otimes L_{\xi}^{\vee}$  in  $K(\mathcal{X})$ , i.e.  $\Psi([V \otimes L_{\xi}^{\vee}]) = G^{\mathcal{V}}(\xi)(\Psi([V]))$ .
- (iii) The pairing  $(\cdot, \cdot)_{\mathcal{V}^{\mathcal{X}}}$  on  $\mathcal{V}^{\mathcal{X}}$  in (37) corresponds to the Mukai pairing on  $K(\mathcal{X})$  defined by  $([V_1], [V_2])_{K(\mathcal{X})} := \chi(V_2^{\vee} \otimes V_1)$ , i.e.  $(\Psi([V_1]), \Psi([V_2]))_{\mathcal{V}^{\mathcal{X}}} = ([V_1], [V_2])_{K(\mathcal{X})}$ . In particular, the pairing  $(\cdot, \cdot)_{\mathcal{V}^{\mathcal{X}}}$  restricted on  $\mathcal{V}_{\mathbb{Z}}^{\mathcal{X}}$  takes values in  $\mathbb{Z}$ .

Therefore  $\mathcal{V}_{\mathbb{Z}}^{\mathcal{X}}$  and  $\mathcal{V}_{\mathbb{R}}^{\mathcal{X}} := \mathcal{V}_{\mathbb{Z}}^{\mathcal{X}} \otimes_{\mathbb{Z}} \mathbb{R}$  satisfy the conditions in Proposition 3.7 except for the unimodularity of the pairing on  $\mathcal{V}_{\mathbb{Z}}^{\mathcal{X}}$ . We call  $\mathcal{V}_{\mathbb{Z}}^{\mathcal{X}}$  and  $\mathcal{V}_{\mathbb{R}}^{\mathcal{X}}$  the  $\widehat{\Gamma}$ -integral structure and the  $\widehat{\Gamma}$ -real structure respectively. The real involution  $\kappa_{\mathcal{V}}$  on  $\mathcal{V}^{\mathcal{X}}$  for the  $\widehat{\Gamma}$ -real structure is given by

$$\kappa_{\mathcal{V}}(\alpha) = (-1)^k \prod_{0 \leq f < 1} \prod_{i=1}^{l_{\mathrm{inv}(v),f}} \frac{\Gamma(1 - f + \delta_{\mathrm{inv}(v),f,i})}{\Gamma(1 - \bar{f} - \delta_{\mathrm{inv}(v),f,i})} \mathrm{inv}^* \bar{\alpha}, \quad \alpha \in H^{2k}(\mathcal{X}_v) \subset \mathcal{V}^{\mathcal{X}},$$

where  $\delta_{\mathrm{inv}(v),f,i}$ ,  $i = 1, \dots, l_{\mathrm{inv}(v),f}$  are the Chern roots of  $(\mathrm{pr}^* T\mathcal{X})_{\mathrm{inv}(v),f}$  and

$$\bar{f} := \begin{cases} 1 - f & \text{if } 0 < f < 1 \\ 0 & \text{if } f = 0. \end{cases}$$

Therefore, this  $\kappa_{\mathcal{V}}$  satisfies (47) and (49). In particular, the conclusions of Theorem 3.9 hold for the  $\widehat{\Gamma}$ -real structure on the quantum cohomology  $\frac{\infty}{2}$  VHS.

The unimodularity of the pairing on  $\mathcal{V}_{\mathbb{Z}}^{\mathcal{X}}$  (or on the integral local system  $R_{\mathbb{Z}}$ ) holds if the map

$$K(\mathcal{X}) \rightarrow \mathrm{Hom}(K(\mathcal{X}), \mathbb{Z}) \quad \alpha \mapsto \chi(\alpha \otimes \cdot)$$

is surjective. This holds true when  $\mathcal{X}$  is a manifold  $X$ . The author does not know if this holds in general.

**Remark 3.17.** Let  $\mathcal{X} = X$  be a manifold. A possible origin of the  $\widehat{\Gamma}$ -class might be the Floer theory on the free loop space  $LX = C^\infty(S^1, X)$ . Let  $S^1$  act on  $LX$  by loop rotation. Givental's heuristic interpretation [17] of the quantum  $D$ -module as the  $S^1$ -equivariant Floer theory suggests that the set  $X$  of constant loops in  $LX$  contributes to the Floer theory by the (infinite) localization factor:

$$\frac{1}{\text{Euler}_{S^1}(N_+)} = \frac{1}{\prod_{m>0} \text{Euler}_{S^1}(TX \otimes \varrho^m)}$$

where  $N_+ \cong \bigoplus_{m>0} TX \otimes \varrho^m$  is the *positive normal bundle* of  $X$  in  $LX$  and  $\varrho$  is the one-dimensional  $S^1$ -module of weight 1. By the  $\zeta$ -function regularization, this factor gives exactly  $z^{-\mu} z^\rho (2\pi)^{-n/2} \widehat{\Gamma}_X$ , where  $z = c_1^{S^1}(\varrho)$  is a generator of  $H^2(BS^1) = H_{S^1}(\text{pt})$ .

#### 4. EXAMPLE: $tt^*$ -GEOMETRY OF $\mathbb{P}^1$

We calculate the Cecotti-Vafa structure on quantum cohomology of  $\mathbb{P}^1$  with respect to the  $\widehat{\Gamma}$ -real structure in Definition-Proposition 3.16. By [24, Theorem 4.11], the  $\widehat{\Gamma}$ -real structure here matches with a natural real structure on the mirror, so the  $tt^*$ -geometry of  $\mathbb{P}^1$  is the same as that of the Landau-Ginzburg model (mirror of  $\mathbb{P}^1$ ):

$$W_q: \mathbb{C}^* \rightarrow \mathbb{C}, \quad W_q = x + \frac{q}{x}, \quad q \in \mathbb{C}^*.$$

Let  $\omega \in H^2(\mathbb{P}^1)$  be the unique integral Kähler class. Let  $\{t^0, t^1\}$  be the linear coordinate system on  $H^*(\mathbb{P}^1)$  dual to the basis  $\{\mathbf{1}, \omega\}$ . Put  $\tau = t^0 \mathbf{1} + t^1 \omega$ . The quantum product  $\circ_\tau$  is given by

$$(\mathbf{1} \circ_\tau) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\omega \circ_\tau) = \begin{bmatrix} 0 & e^{t^1} \\ 1 & 0 \end{bmatrix},$$

where we identify  $\mathbf{1}, \omega$  with column vectors  $[1, 0]^T$ ,  $[0, 1]^T$  and the matrices act on vectors by the left multiplication. The exponential  $e^{t^1}$  corresponds to  $q$  in the Landau-Ginzburg model via the mirror map, so we set  $q = e^{t^1}$ . Hereafter, we restrict  $\tau$  to lie on  $H^2(\mathbb{P}^1)$  but we will not lose any information by this (see Remark 4.1 below). Recall that the Hodge structure  $\mathbb{F}_\tau$  associated with the quantum cohomology of  $\mathbb{P}^1$  is given by the image of  $\mathbb{J}_\tau: H^*(\mathbb{P}^1) \otimes \mathcal{O}(\mathbb{C}) \rightarrow \mathcal{H}^{\mathbb{P}^1} = H^*(\mathbb{P}^1) \otimes \mathcal{O}(\mathbb{C}^*)$  in (38). The  $J$ -function  $J(q, z) = \mathbb{J}_\tau \mathbf{1}$  is given by [18]:

$$J(q, z) := e^{t^1 \omega / z} \sum_{k=0}^{\infty} \frac{q^k \mathbf{1}}{(\omega + z)^2 \cdots (\omega + kz)^2} = e^{t^1 \omega / z} (J_0(q, z) \mathbf{1} + J_1(q, z) \frac{\omega}{z}),$$

and the map  $\mathbb{J}_\tau$  is given by

$$\mathbb{J}_\tau = \begin{bmatrix} \begin{smallmatrix} | \\ J(e^{t^1}, z) \\ | \end{smallmatrix} & \begin{smallmatrix} | \\ z \partial_1 J(e^{t^1}, z) \\ | \end{smallmatrix} \end{bmatrix} = e^{t^1 \omega / z} \circ Q, \quad Q := \begin{bmatrix} J_0 & z \partial_1 J_0 \\ J_1 / z & J_0 + \partial_1 J_1 \end{bmatrix}$$

where  $\partial_1 = (\partial / \partial t^1)$ . By Definition-Proposition 3.16, an integral basis of  $\mathcal{V}^{\mathbb{P}^1} = H^*(\mathbb{P}^1)$  is given by

$$\Psi(\mathcal{O}_{\mathbb{P}^1}) = \frac{1}{\sqrt{2\pi}}(\mathbf{1} - 2\gamma\omega), \quad \Psi(\mathcal{O}_{\text{pt}}) = \sqrt{2\pi} \mathbf{i}\omega,$$

where  $\gamma$  is the Euler constant. Hence the real involutions on  $\mathcal{V}^{\mathbb{P}^1}$  and  $\mathcal{H}^{\mathbb{P}^1}$  are given respectively by (see (45)):

$$\kappa_{\mathcal{V}} = \begin{bmatrix} 1 & 0 \\ -4\gamma & -1 \end{bmatrix} \circ \overline{\phantom{x}}, \quad \kappa_{\mathcal{H}} = \begin{bmatrix} z & 0 \\ -4\gamma & -z^{-1} \end{bmatrix} \circ \overline{\phantom{x}}.$$

where  $\overline{\phantom{x}}$  is the usual complex conjugation (when  $z$  lies in  $S^1 = \{|z| = 1\}$ ).

To obtain the Cecotti-Vafa structure, we need to find a basis of  $\mathbb{F}_{\tau} \cap \kappa_{\mathcal{H}}(\mathbb{F}_{\tau})$ . The procedure below follows the proof of Theorem 3.9 in Section 3.5. Put  $\mathbb{F}'_{\tau} := e^{-t^1\omega/z}\mathbb{F}_{\tau}$  and  $\kappa_{\mathcal{H}}^{\tau} := e^{-(t^1+\overline{t^1})\omega/z}\kappa_{\mathcal{H}}$ . By

$$\mathbb{F}_{\tau} \cap \kappa_{\mathcal{H}}(\mathbb{F}_{\tau}) = e^{t^1\omega/z}(\mathbb{F}'_{\tau} \cap \kappa_{\mathcal{H}}^{\tau}(\mathbb{F}'_{\tau})),$$

it suffices to calculate a basis of  $\mathbb{F}'_{\tau} \cap \kappa_{\mathcal{H}}^{\tau}(\mathbb{F}'_{\tau})$ . First we approximate  $\mathbb{F}'_{\tau}$  by  $\mathbb{F}_{\text{lim}} := H^*(\mathbb{P}^1) \otimes \mathcal{O}(\mathbb{C})$  and solve for a basis of  $\mathbb{F}_{\text{lim}} \cap \kappa_{\mathcal{H}}^{\tau}(\mathbb{F}_{\text{lim}})$ . By elementary linear algebra, we find the following Birkhoff factorization of  $[\kappa_{\mathcal{H}}^{\tau}(\mathbf{1}), \kappa_{\mathcal{H}}^{\tau}(\omega)]$ :

$$[\kappa_{\mathcal{H}}^{\tau}(\mathbf{1}), \kappa_{\mathcal{H}}^{\tau}(\omega)] = BC, \quad B := \begin{bmatrix} 1 & z/a_{\tau} \\ 0 & 1 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & 1/a_{\tau} \\ a_{\tau} & -1/z \end{bmatrix},$$

where  $a_{\tau} := -t^1 - \overline{t^1} - 4\gamma$ . Then the column vectors of  $B$  give a basis of  $\mathbb{F}_{\text{lim}} \cap \kappa_{\mathcal{H}}^{\tau}(\mathbb{F}_{\text{lim}})$  (cf. (15)). Note that the column vectors of  $Q$  above form a basis of  $\mathbb{F}'_{\tau}$ . Thus the Birkhoff factorization of  $Q^{-1}\kappa_{\mathcal{H}}^{\tau}(Q)$  calculates a basis of  $\mathbb{F}'_{\tau} \cap \kappa_{\mathcal{H}}^{\tau}(\mathbb{F}'_{\tau})$ . Define a matrix  $S$  by

$$\kappa_{\mathcal{H}}^{\tau}(Q) = QBS C.$$

Using the fact that  $Q^{-1}$  is the adjoint of  $Q(-z)$  (by Proposition 3.5), we have

$$S = \begin{bmatrix} 2\Re(J_0\overline{J_1})a_{\tau}^{-1} + |J_0|^2 + 2\Re(\partial_1 J_0\overline{J_1} + J_0\overline{\partial_1 J_1}) & (2\Re(J_0\overline{J_1})a_{\tau}^{-2} + (\partial_1 J_0\overline{J_1} + \overline{J_0}\partial_1 J_1)a_{\tau}^{-1} - \partial_1 J_0\overline{J_0}z) \\ + 2\Re(\partial_1 J_0\overline{\partial_1 J_1})a_{\tau} - |\partial_1 J_0|^2 a_{\tau}^2 & \\ (-2\Re(J_0\overline{J_1}) - (\overline{\partial_1 J_0}J_1 + J_0\overline{\partial_1 J_1})a_{\tau} & - 2\Re(J_1\overline{J_0})a_{\tau}^{-1} + |J_0|^2 \\ + J_0\overline{\partial_1 J_0}a_{\tau}^2)z^{-1} & \end{bmatrix},$$

where we restrict  $z$  to lie on  $S^1 = \{|z| = 1\}$ . Because  $S = \mathbf{1} + O(|q|^{1-\epsilon})$ ,  $0 < \epsilon < 1$  as  $|q| \rightarrow 0$ , this admits the Birkhoff factorization  $S = \tilde{B}\tilde{C}$  for  $|q| \ll 1$ , where  $\tilde{B}: \mathbb{D}_0 \rightarrow GL_2(\mathbb{C})$ ,  $\tilde{C}: \mathbb{D}_{\infty} \rightarrow GL_2(\mathbb{C})$  such that  $\tilde{B}(0) = \mathbf{1}$ . Then the column vectors of  $QB\tilde{B} = \kappa_{\mathcal{H}}^{\tau}(Q)C^{-1}\tilde{C}^{-1}$  give a basis of  $\mathbb{F}'_{\tau} \cap \kappa_{\mathcal{H}}^{\tau}(\mathbb{F}'_{\tau})$ . We can perform the Birkhoff factorization in the following way. Note that  $S$  is expanded in a power series in  $q$  and  $\overline{q}$  with coefficients in Laurent polynomials in  $a_{\tau}$  and  $z$ :

$$S = \sum_{n,m \geq 0} S_{n,m} q^n \overline{q}^m, \quad S_{n,m} \in \text{End}(\mathbb{C}^2)[z, z^{-1}, a_{\tau}, a_{\tau}^{-1}].$$

We put  $\tilde{B} = \sum_{n,m \geq 0} \tilde{B}_{n,m} q^n \overline{q}^m$ ,  $\tilde{C} = \sum_{n,m \geq 0} \tilde{C}_{n,m} q^n \overline{q}^m$ . Since  $S_{0,0} = \tilde{B}_{0,0} = \tilde{C}_{0,0} = \text{id}$ , we can recursively solve for  $\tilde{B}_{n,m}$  and  $\tilde{C}_{n,m}$  by decomposing

$$\tilde{B}_{n,m} + \tilde{C}_{n,m} = S_{n,m} - \sum_{(i,j) \neq 0, (n-i, m-j) \neq 0} \tilde{B}_{i,j} \tilde{C}_{n-i, m-j}$$

into strictly positive power series  $\tilde{B}_{n,m}$  and non-positive power series  $\tilde{C}_{n,m}$  in  $z$ . The first six terms of  $B\tilde{B}$  are given by

$$\begin{aligned} B\tilde{B} = & \begin{bmatrix} 1 & \frac{z}{a_\tau} \\ 0 & 1 \end{bmatrix} + \bar{q} \begin{bmatrix} (1+a_\tau)z^2 & \frac{z^3}{a_\tau} \\ (2+2a_\tau+a_\tau^2)z & \frac{(2+a_\tau)z^2}{a_\tau} \end{bmatrix} + q\bar{q} \begin{bmatrix} 0 & -\frac{(8+8a_\tau+2a_\tau^2)z}{a_\tau^2} \\ 0 & 0 \end{bmatrix} \\ & + \bar{q}^2 \begin{bmatrix} \frac{(1+2a_\tau)z^4}{4} & \frac{z^5}{4a_\tau} \\ \frac{(3+6a_\tau+2a_\tau^2)z^3}{4} & \frac{(3+a_\tau)z^4}{4a_\tau} \end{bmatrix} + q\bar{q}^2 \begin{bmatrix} \frac{(33+34a_\tau+18a_\tau^2+4a_\tau^3)z^2}{4} & -\frac{(32+31a_\tau+12a_\tau^2+2a_\tau^3)z^3}{4a_\tau^2} \\ \frac{(25+50a_\tau+34a_\tau^2+12a_\tau^3+2a_\tau^4)z}{2} & -\frac{(64+78a_\tau+45a_\tau^2+14a_\tau^3+2a_\tau^4)z^2}{4a_\tau^2} \end{bmatrix} \\ & + \bar{q}^3 \begin{bmatrix} \frac{(1+3a_\tau)z^6}{36} & \frac{z^7}{36a_\tau} \\ \frac{(11+33a_\tau+9a_\tau^2)z^5}{108} & \frac{(11+3a_\tau)z^6}{108a_\tau} \end{bmatrix} + O((\log |q|)^5 |q|^4) \end{aligned}$$

Let  $\Phi_\tau$  denote the inverse to the natural projection  $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) \rightarrow \mathbb{F}_\tau/z\mathbb{F}_\tau = H^*(\mathbb{P}^1)$ . Because  $B\tilde{B} = \mathbf{1} + O(z)$ , we have  $[\Phi_\tau(\mathbf{1}), \Phi_\tau(\omega)] = e^{t^1\omega/z} QB\tilde{B}$ :

$$\Phi_\tau : H^*(\mathbb{P}^1) = \mathbb{F}'_\tau/z\mathbb{F}'_\tau \xrightarrow{QB\tilde{B}} \mathbb{F}'_\tau \cap \kappa'_{\mathcal{H}}(\mathbb{F}'_\tau) \xrightarrow{e^{t^1\omega/z}} \mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau).$$

The Cecotti-Vafa structure for  $\mathbb{P}^1$  is defined on the trivial vector bundle  $K := H^*(\mathbb{P}^1) \times H^*(\mathbb{P}^1) \rightarrow H^*(\mathbb{P}^1)$ . Recall that the Hermitian metric  $h$  on  $K_\tau$  is the pull-back of the Hermitian metric  $(\alpha, \beta) \mapsto (\kappa_{\mathcal{H}}(\alpha), \beta)_{\mathcal{H}}$  on  $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$  through  $\Phi_\tau : K_\tau \cong \mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$ . The Hermitian metric  $h$  is of the form:

$$h = \begin{bmatrix} h_{\bar{0}0} & 0 \\ 0 & h_{00}^{-1} \end{bmatrix}, \quad h_{\bar{0}0} := \int_{\mathbb{P}^1} \kappa_{\mathcal{H}}(\Phi_\tau(\mathbf{1})) \Big|_{z \mapsto -z} \cup \Phi_\tau(\mathbf{1}).$$

The first seven terms of the expansion of  $h_{\bar{0}0}$  are (with  $a_\tau = -t^1 - \bar{t}^1 - 4\gamma$ ,  $q = e^{t^1}$ )

$$\begin{aligned} h_{\bar{0}0} = & a_\tau + |q|^2 (a_\tau^3 + 4a_\tau^2 + 8a_\tau + 8) + |q|^4 \left( a_\tau^5 + 8a_\tau^4 + \frac{121}{4}a_\tau^3 + \frac{129}{2}a_\tau^2 + \frac{145}{2}a_\tau + \frac{145}{4} \right) \\ & + |q|^6 \left( a_\tau^7 + 12a_\tau^6 + \frac{275}{4}a_\tau^5 + \frac{477}{2}a_\tau^4 + \frac{9539}{18}a_\tau^3 + \frac{81001}{108}a_\tau^2 + \frac{50342}{81}a_\tau + \frac{55526}{243} \right) \\ & + |q|^8 \left( a_\tau^9 + 16a_\tau^8 + \frac{493}{4}a_\tau^7 + \frac{1185}{2}a_\tau^6 + \frac{31001}{16}a_\tau^5 + \frac{79939}{18}a_\tau^4 + \frac{49077907}{6912}a_\tau^3 \right. \\ & \quad \left. + \frac{52563371}{6912}a_\tau^2 + \frac{614694323}{124416}a_\tau + \frac{736622003}{497664} \right) \\ & + |q|^{10} \left( a_\tau^{11} + 20a_\tau^{10} + \frac{775}{4}a_\tau^9 + \frac{2381}{2}a_\tau^8 + \frac{368599}{72}a_\tau^7 + \frac{1738481}{108}a_\tau^6 + \frac{780126811}{20736}a_\tau^5 \right. \\ & \quad \left. + \frac{4053627445}{62208}a_\tau^4 + \frac{254355946241}{3110400}a_\tau^3 + \frac{1465574917127}{20736000}a_\tau^2 + \frac{163291639271}{4320000}a_\tau + \frac{1840366543439}{194400000} \right) \\ & + |q|^{12} \left( a_\tau^{13} + 24a_\tau^{12} + \frac{1121}{4}a_\tau^{11} + \frac{4193}{2}a_\tau^{10} + \frac{1606399}{144}a_\tau^9 + \frac{2398517}{54}a_\tau^8 + \frac{2814667745}{20736}a_\tau^7 + \frac{20004983519}{62208}a_\tau^6 \right. \\ & \quad \left. + \frac{407437321759}{691200}a_\tau^5 + \frac{51278023471273}{62208000}a_\tau^4 + \frac{796478452045403}{933120000}a_\tau^3 + \frac{11553263487112967}{18662400000}a_\tau^2 \right. \\ & \quad \left. + \frac{11823418405646927}{41990400000}a_\tau + \frac{15268380040196927}{251942400000} \right) + \dots \end{aligned}$$

The other data  $(\kappa, g, C, \tilde{C}, D, \mathcal{U}, \overline{\mathcal{U}}, \mathcal{Q})$  of the Cecotti-Vafa structure are given in terms of  $h_{\overline{0}0}$ . In fact, we have  $C_0 = \tilde{C}_0 = \text{id}$ ,  $D_0 = \partial/\partial t^0$ ,  $D_{\overline{0}} = \partial/\partial \overline{t}^0$  and

$$\begin{aligned} g &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \kappa = \begin{bmatrix} 0 & h_{\overline{0}0}^{-1} \\ h_{\overline{0}0} & 0 \end{bmatrix} \circ -, \quad D_1 = \partial_1 + \begin{bmatrix} \partial_1 \log h_{\overline{0}0} & 0 \\ 0 & -\partial_1 \log h_{\overline{0}0} \end{bmatrix}, \\ D_{\overline{1}} &= \overline{\partial}_1, \quad C_1 = \frac{1}{2}\mathcal{U} = \begin{bmatrix} 0 & e^{t^1} \\ 1 & 0 \end{bmatrix}, \quad \tilde{C}_{\overline{1}} = \frac{1}{2}\overline{\mathcal{U}} = \begin{bmatrix} 0 & h_{\overline{0}0}^{-2} \\ e^{\overline{t}^1} h_{\overline{0}0}^2 & 0 \end{bmatrix}, \\ \mathcal{Q} &= \partial_E + \mu - D_E = \begin{bmatrix} -\frac{1}{2} - 2\partial_1 \log h_{\overline{0}0} & 0 \\ 0 & \frac{1}{2} + 2\partial_1 \log h_{\overline{0}0} \end{bmatrix}, \end{aligned}$$

where  $\partial, \overline{\partial}$  are the connections given by the given trivialization of  $K$ .

**Remark 4.1.** (i) Takahashi [37] classified the  $tt^*$ -geometry of rank 2. The  $\mathbb{P}^1$  case is included in the consideration in Section 5 *ibid.*, but this does not seem to appear in Theorem 5.1 *ibid.* It is shown in Lemma 2.1 *ibid.* that the Hermitian metric  $h$  is represented by a diagonal matrix with determinant 1.

(ii) From the theory of (trTERP)+(trTLEP) structure on the tangent bundle, it follows that  $h, C, \tilde{C}, \mathcal{U}, \overline{\mathcal{U}}, \mathcal{Q}$  are invariant under the flow of the unit vector field  $(\partial/\partial t^0), (\partial/\partial \overline{t}^0)$ . Therefore, the calculation here determines the Cecotti-Vafa structure on the big quantum cohomology. Moreover we have  $D_E + \mathcal{Q} = \partial_E + \mu$  and  $\text{Lie}_{E-\overline{E}} h = 0$ . In the case of  $\mathbb{P}^1$ , this means that  $h_{\overline{0}0}$  depends only on  $|q|$ . See [21].

(iii) We can show that our procedure for the Birkhoff factorization gives convergent series for sufficiently small values of  $|q|$ . In particular, the expansion for  $h_{\overline{0}0}$  converges for small  $|q|$ .

(iv) Since the preprint version [23] of this paper was written, Dorfmeister-Guest-Rossman [14] found that the  $tt^*$ -geometry of  $\mathbb{P}^1$  gives a new example of a CMC surface in Minkowski space  $\mathbb{R}^{2,1}$ .

We explain a different way to calculate the Hermitian metric  $h_{\overline{0}0}$  due to Cecotti-Vafa [6]. The  $tt^*$ -equation  $[D_1, D_{\overline{1}}] + [C_1, \tilde{C}_{\overline{1}}] = 0$  (see Proposition 2.13) gives the following differential equation for  $h_{\overline{0}0}$ :

$$(61) \quad \partial_1 \overline{\partial}_1 \log h_{\overline{0}0} = -h_{\overline{0}0}^{-2} + |q|^2 h_{\overline{0}0}^2.$$

Cecotti-Vafa [6] identified  $h_{\overline{0}0}$  with a unique solution to (61) expanded in the form

$$h_{\overline{0}0} = \sum_{n=0}^{\infty} F_n |q|^{2n}, \quad F_0 = a_\tau, \quad F_n \in \mathbb{C}[a_\tau, a_\tau^{-1}], \quad a_\tau = -2 \log |q| - 4\gamma.$$

The equation (61) gives an infinite set of recursive differential equations for  $F_n$ . It is easy to check that the differential equations determine the Laurent polynomial  $F_n$  *uniquely*. Moreover it turns out that  $F_n \in \mathbb{Q}[a_\tau]$  and  $\deg F_n = 2n + 1$ . The *existence* of such a solution seems to be non-trivial, but the Birkhoff factorization certainly gives such  $h_{\overline{0}0}$ . By physical arguments, Cecotti-Vafa [5, 8, 6] showed that  $h_{\overline{0}0}$  should be positive and smooth on the positive real axis  $0 < |q| < \infty^5$ . Since the Landau-Ginzburg mirror of  $\mathbb{P}^1$  is given by a cohomologically tame function, this follows from

<sup>5</sup> For this, the constant  $\gamma$  in  $a_\tau$  must be the very Euler constant.

the Sabbah's result [34] in singularity theory (see Remark 3.12). Therefore, the Cecotti-Vafa structure for  $\mathbb{P}^1$  is well-defined and positive definite on the whole  $H^*(\mathbb{P}^1)$ .

Cecotti-Vafa [5] also found that the differential equation (61) is equivalent to the Painlevé III equation:

$$\frac{d^2 u}{dz^2} + \frac{1}{z} \frac{du}{dz} = 4 \sinh(u), \quad h_{\overline{0}0} = e^{u/2} |e^{-t^1/2}|, \quad z = 4|e^{t^1/2}|.$$

It seems that the solution corresponding to our  $h_{\overline{0}0}$  has been obtained in the study of Painlevé III equation [25, 30] (in fact, the first few terms of the expansion of their solutions match with ours). If this is the case,  $h_{\overline{0}0}$  should have the asymptotics [25, 30] (also appearing in [6]):

$$h_{\overline{0}0} \sim \frac{1}{\sqrt{|q|}} \left( 1 - \frac{1}{2\sqrt{\pi}|q|^{1/4}} e^{-8|q|^{1/2}} \right)$$

as  $|q| \rightarrow \infty$ . With respect to the metric  $h_{\overline{1}1} = h_{\overline{0}0}^{-1}$  on the Kähler moduli space  $H^2(\mathbb{P}^1)/2\pi i H^2(\mathbb{P}^1, \mathbb{Z})$ , a neighborhood of the large radius limit point  $q = 0$  has negative curvature, but does not have finite volume. The curvature  $-\frac{2}{h_{\overline{0}0}}(1 - |q|^2 h_{\overline{0}0}^4)$  goes to zero as  $|q| \rightarrow 0$  and  $|q| \rightarrow \infty$  and the total curvature is  $-\pi/4$ . Much more examples including  $\mathbb{P}^n$ ,  $\mathbb{P}^1/\mathbb{Z}_n$  are calculated in physics literature. We refer the reader to [5, 6, 7].

## 5. APPENDIX

**5.1. Proof of (58).** Birkhoff's theorem implies that there exists an open dense neighborhood of  $\mathbf{1}$  in the loop group  $LGL_N(\mathbb{C})$  which is diffeomorphic to the product of subgroups  $L_1^+ GL_N(\mathbb{C}) \times L^- GL_N(\mathbb{C})$  [33]. We use the inverse function theorem for Hilbert manifolds to explain the order estimate in (58). Consider the space  $LGL_N(\mathbb{C})^{1,2}$  of Sobolev loops which consists of maps  $\lambda: S^1 \rightarrow GL_N(\mathbb{C})$  such that  $\lambda$  and its weak derivative  $\lambda'$  are square integrable. Note that this is a subgroup of the group of continuous loops by Sobolev embedding theorem  $W^{1,2}(S^1) \subset C^0(S^1)$  and the multiplication theorem  $W^{1,2}(S^1) \times W^{1,2}(S^1) \rightarrow W^{1,2}(S^1)$ .  $LGL_N(\mathbb{C})^{1,2}$  is a Hilbert manifold modeled on the Hilbert space  $W^{1,2}(S^1, \mathfrak{gl}_N(\mathbb{C}))$ . A co-ordinate chart of a neighborhood of  $\mathbf{1}$  is given by the exponential map  $A(z) \mapsto e^{A(z)}$ . Let  $L_1^+ GL_N(\mathbb{C})^{1,2}$  be the subgroup of  $LGL_N(\mathbb{C})^{1,2}$  consisting of the boundary values of holomorphic maps  $\lambda_+: \{|z| < 1\} \rightarrow GL_N(\mathbb{C})$  satisfying  $\lambda_+(0) = \mathbf{1}$ . Let  $L^- GL_N(\mathbb{C})^{1,2}$  be the subgroup of  $LGL_N(\mathbb{C})^{1,2}$  consisting of the boundary values of holomorphic maps  $\lambda_-: \{|z| > 1\} \cup \{\infty\} \rightarrow GL_N(\mathbb{C})$ . Notice that  $W^{1,2} := W^{1,2}(S^1, \mathfrak{gl}_N(\mathbb{C}))$  has the direct sum decomposition:

$$(62) \quad W^{1,2} = W_+^{1,2} \oplus W_-^{1,2},$$

where  $W_+^{1,2}$  ( $W_-^{1,2}$ ) is the closed subspace of  $W^{1,2}(S^1, \mathfrak{gl}_N(\mathbb{C}))$  consisting of strictly positive Fourier series  $\sum_{n>0} a_n z^n$  (non-positive Fourier series  $\sum_{n\leq 0} a_n z^n$  resp.) with  $a_n \in \mathfrak{gl}_N(\mathbb{C})$ . The subgroups  $L_1^+ GL_N(\mathbb{C})^{1,2}$  and  $L^- GL_N(\mathbb{C})^{1,2}$  are modeled on the Hilbert spaces  $W_+^{1,2}$  and  $W_-^{1,2}$  respectively. Consider the multiplication map  $L_1^+ GL_N(\mathbb{C})^{1,2} \times L^- GL_N(\mathbb{C})^{1,2} \rightarrow LGL_N(\mathbb{C})^{1,2}$ . The differential of this map at the identity is given by the sum  $W_+^{1,2} \times W_-^{1,2} \rightarrow W^{1,2}$  and is clearly an isomorphism. By the inverse function

theorem for Hilbert manifolds, there exists a differentiable inverse map on a neighborhood of  $\mathbf{1}$ . In the case at hand, we have  $\|(B_t^{-1}Q_tB_t)(C_t\bar{Q}_tC_t^{-1}) - \mathbf{1}\|_{W^{1,2}} = O(e^{-\epsilon t})$  as  $t \rightarrow \infty$ . Therefore, this admits the Birkhoff factorization (58) for  $t \gg 0$  with  $\|\tilde{B}_t - \mathbf{1}\|_{W^{1,2}} = O(e^{-\epsilon t})$  and  $\|\tilde{C}_t - \mathbf{1}\|_{W^{1,2}} = O(e^{-\epsilon t})$ . By Sobolev embedding, the order estimates hold also for the  $C^0$ -norm. (The method here does not work directly for the Banach manifold of continuous loops, since the decomposition (62) is not true in this case.)

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